A Quick Note on Calculating the Radius of Convergence

The **radius of convergence** is a number ρ such that the series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges absolutely for $|x - x_0| < \rho$, and diverges for $|x - x_0| > 0$ (see Fig.1).

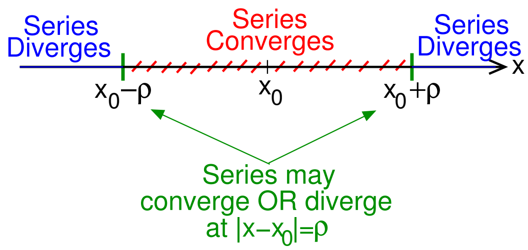

Figure 1: Radius of convergence.

Note that:

- If the series converges ONLY at $x = x_0$, $\rho = 0$.
- If the series converges for ALL values of x, ρ is said to be infinite.

How do we calculate the radius of convergence? Use the **Ratio Test.**

$$\text{Ratio Test}: \sum_{n=0}^{\infty} b_n \text{ converges if } \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| < 1.$$

So

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges for x such that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| < 1 \Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x - x_0| < 1.$$

EXAMPLE: Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{n2^n}.$$

To find the radius of convergence, use the ratio test:

$$1 > \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}/((n+1)2^{n+1})}{(x+1)^n/(n2^n)} \right| = \lim_{n\to\infty} \left| \left(\frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \right) \left(\frac{n2^n}{(x+1)^n} \right) \right| = \lim_{n\to\infty} \left| \frac{n}{2(n+1)} \right| |x+1|$$

$$\Rightarrow 1 > \frac{1}{2}|x+1|.$$

Thus, the series converges absolutely for $|x + 1| < 2$ or $-3 < x < 1$, and diverges for $|x + 1| > 2$. The radius of convergence about $x_0 = -1$ (recall the general series is in terms of $(x - x_0)^n$) is $\rho = 2$.
Left for students: what can you say about convergence at the endpoints?

Alternatively, we can exploit the singularities! If the series is a Taylor series of some function, f, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \ ,$$

where $a_n = \frac{f^{(n)}(x_0)}{n!}$, then the radius of convergence is equal to the distance between x_0 and the singularity of f that is closest to x_0 in the complex plane, as long as the function f is sufficiently "nice". The desired notion of "niceness" is beyond what can be stated here but is found in most standard complex variables textbooks. Most functions you are familar with will work, e.g. e^x, $\sin(x)$, $\frac{1}{1-x}$ and any polynomial are "nice".[1]
A singularity is any point where the function is not defined.

EXAMPLE: Consider

$$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} a_n (x - x_0)^n \ .$$

The singularities of f are where $1 + x^2 = 0$, i.e. $x = \pm i$. We look at the distance between x_0 and these singularies. Assuming $x_0 \in \mathbb{R}$ the distance to each is the same so lets compute the distance to i. This distance is $|x_0 - i| = \sqrt{x_0^2 + 1}$. So if $x_0 = 0$, the radius of convergence of the above series is $\sqrt{0+1} = 1$. If $x_0 = 2$, the radius of convergence is $\sqrt{5}$ (so converges in $(2 - \sqrt{5}, 2 + \sqrt{5})$).

[1] An exception is $h(x) = e^{-x^{-2}}$. Though strictly not defined at $x = 0$, as $x \to 0$, $h(x) \to 0$. In fact as $x \to 0$, $h^{(n)}(x) \to 0$, for every positive integer n and so the Taylor series of h centred at $x = 0$ would just be zero. Another exception is $h(x) = |x|$.