Math 257/316 Assignment 7, Written Part Due Monday Nov. 3 rd in class

Note: The final Exam is on Friday Dec the 5 th at 8:30 am.

Problem 1: (Old Exam Question) Solve the initial boundary value problem:

$$u_t = u_{xx} - u - xe^{-t} \quad 0 < x < 1,\ t > 0$$
$$\text{BC: } u(0,t) = 0, \quad u(1,t) = 0$$
$$\text{IC: } u(x,0) = x$$

Problem 2: (A bar subject to an external heat source). Consider a bar of length π and thermal diffusivity 1 having initial temperature distribution given by $u(x,0) = \cos(\frac{3x}{2})$. Suppose that the left end is insulated and that the right end of the bar is maintained at a temperature of 0 and that a time dependent heat source $e^{-t}\cos(\frac{x}{2})$ is applied across the length of the bar. Thus

$$u_t = u_{xx} + e^{-t}\cos(\frac{x}{2}), \quad 0 < x < \pi,\ t > 0$$
$$\text{BC: } u_x(0,t) = 0 = u(\pi,t)$$
$$\text{IC: } u(x,0) = \cos(\frac{3x}{2})$$

Find the temperature $u(x,t)$.

Problem 3: (A bar subject to a time dependent boundary condition). Consider a bar of length 1 and thermal diffusivity α^2 having initial temperature distribution given by $u(x,0) = \beta \sin 2\pi x$ with $\beta > 0$. Suppose that the left end of the bar is maintained at a temperature of 0 while the temperature of the right end of the bar varies according to $\sin t$. Thus

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \quad 0 < x < 1,\ t > 0 \\ \text{BC: } & u(0,t) = 0,\ u(1,t) = \sin t \\ \text{IC: } & u(x,0) = \beta \sin 2\pi x, \quad \beta > 0 \end{aligned} \tag{1}$$

a) Identify an appropriate function that can be used to reduce the problem with inhomogeneous boundary conditions to one with homogeneous boundary conditions.

b) Use the method of eigenfunction expansions to determine $u(x,t)$.

c) **Stability of the finite difference scheme:** In class we used the discrete Fourier modal representation $u_n^k = \phi_k e^{in\theta}$ to analyze the stability of the finite difference scheme for the heat equation in which thermal diffusivity $\alpha^2 = 1$. Now perform the same analysis for the finite difference scheme:

$$u_n^{k+1} = u_n^k + \alpha^2 \frac{\Delta t}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$$

to derive a time step restriction for the case $\alpha^2 > 0$.

Spread Sheet Projects - to be submitted online using the UBC Vista system before Midnight on the 3 rd Nov.

Problem 3: (EXCEL Part) The sample spreadsheet `HeatFS_TDBC.xls` solves the boundary value problem given in the supplemental problem 4 below. On sheet1 (comparison) the Fourier Series solution is used to determine the solution at the time horizon $t = 0.05$, while on sheet2 (numerical) the corresponding finite difference solution is determined over the time interval $0 < t \leq 0.05$. The graph compares the two solutions at $t = 0.05$. Now adapt the spread sheet `HeatFS_TDBC.xls` (sheet 2) to solve the initial-boundary value problem (1) numerically using finite differences. Set the value of β to be

$$\beta = \frac{\text{your UBC student \#}}{10^6}.$$

Enter the value you found numerically for $u(x = 0.5, t = 1)$ **on Vista. Submit your spreadsheet with your computations as well.** Now code the corresponding Fourier Series solution obtained in part (b) of this question into sheet1 and evaluate $u(x, t)$ at the time horizon $t = 1.5$ and using $\alpha = 1$. Plot the graph comparing the two solutions at $t = 1.5$. For $\Delta x = 0.05$ and $\Delta t = 0.0005$ you will need about 3000 time steps to reach $t = 1.5$! **Submit your spreadsheet including the graph using Vista.**

Stability: Now test the validity of your stability criterion derived in part (c) by performing the following experiments. What happens if we change α^2 to the following values: $\alpha^2 = 2.4,\ 2.5,\ 2.6,\ 3.0$? **Enter your answer on Vista.**

Supplemental solution: Not to be handed in - solution is posted.

Problem 4: (A bar subject to a time dependent boundary condition). Consider a bar of length L and thermal diffusivity α^2 having initial temperature distribution of 0.Suppose that the right end of the bar is maintained at a temperature of 0 while the temperature of the left end of the bar varies according to At. Thus

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \quad 0 < x < 1,\ t > 0 \\ \text{BC: } & u(0,t) = At,\ u(1,t) = 0 \\ \text{IC: } & u(x,0) = 0 \end{aligned} \tag{2}$$

a) Identify an appropriate particular solution to the heat equation that can be used to reduce the problem with inhomogeneous boundary conditions to one with homogeneous boundary conditions.

b) Use the method of eigenfunction expansions to determine $u(x, t)$.

c) Numerics: Solve the initial-boundary value problem (2) numerically using finite differences. Now code the corresponding Fourier Series solution obtained in part (b) of this question and evaluate $u(x, t)$ at the time horizon $t = 0.05$ and using, $L = 1$, $A = 2$, and $\alpha = 1$. Plot the graph comparing the two solutions at $t = 0.05$.Use $\Delta x = 0.05$ and $\Delta t = 0.0005$ for the finite difference solution.(the answer is given in the spreadsheet: `HeatFS_TDBC.xls`.