HOMEWORK 1 SOLUTIONS: MATH 265

Problem 1: In each case, solve for $y(t)$:

- a. $y' + 2y = 0$ with $y(0) = 1$.
- b. $y' + 2y = 3e^t$ with $y(0) = 0$.
- c. $y' + 2y = te^{-2t}$ with $y(1) = 0$.
- d. $ty' + 2y = \cos t$ with $y(\pi) = 0$.
- e. $t^3y' + 4t^2y = e^t$ with $y(1) = 0$.

Solution: Problem 1:

a. Using integrating factor $\mu(t) = \exp(2t)$, or by inspection, the solution is found to be $y = C\exp(-2t)$. Using the initial condition, the solution is $y = \exp(-2t)$.

b. The integrating factor is the same as in part (a). Integrating, we find the solution $y = \exp(t) + C\exp(-2t)$. Using the initial condition, the solution is $y = \exp(t) - \exp(-2t)$.

c. Again the integrating factor is the same. The integral $\mu(t)t\exp(-2t)$ has to be done by parts. The solution is $y = \frac{1}{2}\exp(-2t)t^2 + C\exp(-2t)$ and the solution is $y = \frac{1}{2}\exp(-2t)(t^2 - 1)$.

d. First rewrite the equation in standard form: $y' + (2/t)y = \cos(t)/t$. The integrating factor is found to be $\mu(t) = t^2$. The right hand side integral is therefore $\int t\cos t dt = \cos t + t\sin t + C$. The solution is therefore $y = (\cos t + t\sin t + C)/t^2$. Using the initial condition, we obtain $C = 1$.

e. In standard form, this is $y' + (4/t)y = \exp(t)/t^3$. Similarly to part d, the integrating factor is found to be $\mu(t) = t^4$. The right hand side integral is therefore $\int te^t dt = e^t(t - 1) + C$. Using the initial condition we find the answer $y = e^t(t - 1)/t^4$.

Problem 2: A radioactive substance emits radiation and changes to a new substance. The amount of radioactivity is proportional to the quantity of the original substance remaining. The half-life of a radioactive substance is the amount of time it takes for half the substance to decay radioactively. If the half-life of radium is 1760 years, how much radium is left from an initial gram of radium after 100 years?

Solution: Problem 2: The amount of radioactivity $R = R(t)$, in grams, satisfies

$$R' = -kR\,, \quad R(0) = R_0\,,$$

where $k > 0$ is constant, t is time measured in years, and R_0 is the initial amount of substance. For us, $R(0) = 1$ gram. The solution is

$$R = e^{-kt}R_0\,. \tag{2.1}$$

We are given that $R = R_0/2$, when $t = 1760$ years. This allows us to find k from (2.1) as

$$e^{-1760k} = \ln(1/2) \qquad \rightarrow \qquad k = \ln 2/1760 \approx .000394$$

The units of k are years^{-1}. Now find R when $t = 100$ years. We get from (2.1) with $R_0 = 1$ that

$$R = \exp\left(-\ln 2100/1760\right) \approx .96 \text{ grams}\,.$$

Hence, very little of the original material has dissipated after 100 years.

Problem 3: In a murder investigation a corpse was found by Inspector Clouseau at exactly 8 : 00 p. m. Being alert, he measures the temperature of the body and finds it to be 70°F (Fahrenheit). Two hours later, Inspector Clouseau again measures the temperature of the corpse and finds it to be 60° F. If the room temperature is 50° F, and we assume that Newton's law of cooling applies, when did the murder occur? (Assume that the temperature of the body at the time of the murder was 98.6°).

Solution: Problem 3: Let $T(t)$ be the temperature of the body in degrees Fahrenheit starting at $t = 0$, which measures hours starting from 8 : 00p.m. Then, $T(t)$ satisfies

$$T^{'} = -k(T - 50)\,, \qquad T(0) = 70\,. \tag{4.1}$$

We also know that $T(2) = 60$. The solution to (4.1) is

$$T = 50 + 20e^{-kt}\,. \tag{4.2}$$

Now set $T = 60$, when $t = 2$ to determine k as

$$k = \ln(2)/2 \approx .34657....$$

Now we want to find t such that $T = 98.6$. Clearly, this value of t will satisfy $t < 0$, as the murder occurs before 8 : 00p.m. Thus,

$$98.6 = 50 + 20e^{-kt}\,, \qquad t = -k^{-1}\ln\left(48.6/20\right) = -2.562 \approx -2.6\,.$$

Thus, the murder occurs approximately 156 minutes before 8 : 00p.m. or at 5 : 24p.m.

Problem 4: In the previous problem, Inspector Clouseau concluded that the time of the murder was 2.6 hours before he took the first temperature reading, or at 5 : 24 p. m. However, someone points out that Clouseau's analysis is faulty because the room temperature in which the corpse was found was not constant but instead decreased exponentially according to the law $50e^{-.05t}$, where t is the time (in hours) starting from 8 : 00 p.m. Assume that the room temperature obeys this law.

- (i) What is the differential equation that Inspector Clouseau must solve now?
- (ii) What is the temperature of the body at any time t?
- (iii) When was the time of the murder?

Solution: Problem 4:

- (i) The differential equation is now

$$T^{'} = -k(T - 50e^{-.05t}), \qquad T(0) = 70.$$

- (ii) Multiply by e^{kt} and integrate to get

$$T(t) = \frac{50k}{(k - .05)} e^{-t/20} + ce^{-kt}.$$

Now satisfy $T(0) = 70$, which determines c. Thus,

$$T(t) = \frac{50k}{(k - .05)} e^{-t/20} + \left(70 - \frac{50k}{(k - .05)}\right) e^{-kt}.$$

- (iii) This last part is quite hard. Note that it is difficult to determine k by setting $T(2) = 60$, since we have a nonlinear equation for k to solve. This equation can be written in the form

$$60 - 50e^{-.1} = \frac{3}{k} + 20e^{-2k} - \frac{3.5}{k} e^{-2k}.$$

Using a graphing calculator, or Newton's method, the solution to this equation is $k \approx .29$. Now the murder occurs at a time τ when $T = 98.6$. This Newton's method is used to solve for τ, which gives $\tau = -3.5$. Thus, the murder occurred 210 minutes before 8 : 00p.m. or at 4 : 30p.m.

Problem 5: A spherical raindrop evaporates at a rate proportional to its surface area. If the radius of the raindrop is initially 3 mm and after 1/2 of an hour later it has been reduced to 2 mm, find the time at which the raindrop has completely evaporated.

Solution: Problem 5: The volume of the drop $V = V(t)$ is $V = 4\pi r^3/3$, where $r = r(t)$. The differential equation for the drop is $V^{'} = -kA$, where $A = 4\pi r^2$. Using the chain rule we then get $V^{'} = Ar^{'}$, so that the radius of the raindrop satisfies

$$r^{'} = -kr, \qquad r(0) = 3, \qquad (*)$$

where t is measured in hours and r is measured in mm. We are also given that $r(0.5) = 2$. Solving $(*)$ we get $r = -kt + 3$. Next, $r(0.5) = 2$ gives $k = 2$. Finally, the drop has evaporated when $r = 0$. This occurs when $t = 3/k = 1.5$ hours.

Problem 6: A ball of mass m falls from rest from a height h towards the ground. We assume that the ball is acted upon by a constant gravitational force and by an opposing frictional force, which is proportional to the velocity. Thus, the velocity $v = v(t)$ (with $v > 0$ if the ball is falling downwards) satisfies

$$m\frac{dv}{dt} = mg - kv \,, \qquad v(0) = 0 \,,$$

where $k > 0$ is a constant.

- (i) Calculate the velocity at any later time t before the ball hits the ground.
- (ii) What is the terminal velocity?
- (iii) Give a formula (in terms of an integral) for the time at which the ball hits the ground.

Solution:

- (i) Rewriting the equation in standard form, we have $v' + (k/m)v = g$. Using the integrating factor $\mu(t) = \exp(kt/m)$, we find

  $$v(t) = \frac{gm}{k} + Ce^{-kt/m}.$$

  We use the initial condition $v(0) = 0$ to obtain the solution

  $$v(t) = \frac{gm}{k}\left(1 - e^{-kt/m}\right).$$

- (ii) Letting $t \to \infty$ in the previous formula, we obtain that $v \to v_e = gm/k$, which is the terminal velocity.

- (iii) We have $v = dx/dt$. When $t = 0$, we have $x = 0$, while when $x = h$ we have $t = T$, the time when the ball hits the ground. Integrating $v(t)$, we obtain an implicit equation for T:

  $$h = \int_0^T v(t)dt = \frac{gm}{k}\int_0^T \left(1 - e^{-kt/m}\right) dt$$

  This is the implicit formula that was asked for, however, we can actually do the integral to obtain

  $$h = \frac{gmT}{k} + \frac{gm^2}{k^2}\left(e^{-kT/m} - 1\right)$$

  which is somehow more satisfactory.