HOMEWORK 3

due in class on October 6th

Problem 1: Solve the following initial value problems for $y(x)$:
(a) $y'' - 4y' - 5y = 0, \;\; y(-1) = 3, \;\; y'(-1) = 9.$
(b) $y''' + 2y'' - 5y' - 6y = 0, \;\; y(0) = 2, \;\; y'(0) = 6, \;\; y''(0) = 0.$
(c) $y'' + y = 2e^{-x}, \;\; y(0) = 0, \;\; y'(0) = 0.$
(d) $y'' + 2y' + y = x^2 + 1 - e^x, \;\; y(0) = 0, \;\; y'(0) = 2.$
(e) $y'' - 2y' + y = 8e^x, \;\; y(0) = 3, \;\; y'(0) = 2.$
(f) $y'' + 2y' + 2y = 5\cos(2x), \;\; y(\pi) = -1/2, \;\; y'(\pi) = 1.$

Problem 2: The suspension in a car can be modeled as a vibrating spring with damping due to the shock absorbers. This leads to the equation for the vertical displacement $x(t)$ at time t,

$$mx''(t) + bx'(t) + kx(t) = 0,$$

where m is the mass of the car, b is the damping constant of the shocks, and k is the spring constant. If the mass m of the car is 1000kg and the spring constant k is 3000kg/s^2, determine the minimum value for the damping constant b in kilograms per seconds that will provide a smooth, *oscillation-free* ride. If we replace the springs with heavy-duty ones having twice the spring constant k, how will this minumum change?

Problem 3: A vibrating spring *without* damping can be modeled by the initial value problem:

$$my''(t) + ky(t) = 0 \quad y(0) = y_0, \quad y'(0) = y_1$$

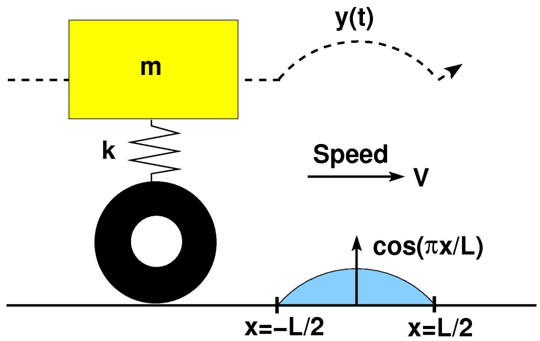
for m the mass of the spring and k is the spring constant.
(a) If $m = 10$kg, $k = 250$kg/s^2, $y_0 = 0.3$m, and $y_1 = -0.1$m/s, find the equation of motion $y(t)$ for this undamped vibrating spring.
(b) What is the frequency of oscillation of this spring system?

Problem 4: A vibrating spring *with* damping can be modeled by the initial value problem:

$$my''(t) + by'(t) + ky(t) = 0 \quad y(0) = y_0, \quad y'(0) = y_1$$

for m the mass of the spring, k is the spring constant, and b the damping constant.
(a) Using the same values for m, k, y_0, and y_1 as in Problem 3, now with $b = 60$kg/s, find the equation of motion $y(t)$ for this damped vibrating spring.
(b) What is the frequency of oscillation of this spring system?
(c) Compare the results of problems 3 and 4 and determine what effect the damping has on the frequency of oscillation. What other effects does it have on the solution? What is the long-time behaviour of the solution (behaviour of the solution as $t \to \infty$)?

Problem 5: Often bumps like the one depicted below are built into roads to discourage speeding.



The figure suggests that a crude model of the vertical motion $y(t)$ of a car encountering the speed bump with speed V is given by:

$$y(t) = 0 \quad \text{for } t \leq -L/2V$$

$$my'' + ky = \begin{cases} \cos(\pi V t/L) & \text{for } -L/2V \leq t \leq L/2V \\ 0 & \text{for } t \geq L/2V \end{cases} .$$

(The absence of a damping term indicates that the car's shock absorbers are broken.) Note that the equations are dependent on time only; as the speed is given as V, we can write space x in terms of time t: $x = Vt$.

(a) Solve this initial value problem; take $m = k = 1$ and $L = \pi$ for convenience. Thus show that the formula for oscillatory motion *after the car has traversed the speed bump* is $y(t) = A\sin(t)$, where A depends on the speed V.

(b) Plot the amplitude $|A|$ of the solution $y(t)$ in part (a) versus the car's speed V. From the graph, estimate the speed that produces the most violent shaking of the vehicle.