Monday: Maximum-minimum problems

1. Of all positive numbers whose sum is 50, find the two that have the maximum product. That is, maximize $Q = xy$, where $x + y = 50$.

2. Minimize $Q = 2x^2 + 3y^2$ where $x + y = 5$

3. The sum of two positive number is 20. Find the numbers (a) if their product is a maximum, (b) if the sum of their squares is a minimum, (c) if the product of the square of one and the cube of the other is a maximum.

4. Consider a 1 km race track with circular ends. Find the values of x (length of the straight sides) and y (width; twice the radius of the circular ends) that will maximize the area of the rectangle.

Tuesday-Friday: Maximum-minimum and related rates problems

1. A spherical piece of ice melts so that its surface area decreases at a rate of 1 cm^2/min. Find the rate that the diameter decreases when the diameter is 5 cm.

2. Find the rectangle with the largest area that can be placed inside the region between the curve $y = 4 - x^2$ and the horizontal axis $y = 0$.

3. A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at the rate of 0.5 m/min, how fast is the top of the ladder sliding down the wall when the base of the ladder is 1 m away from the wall?

4. Water is leaking out of a small hole at the tip of a conical paper cup at the rate of 1 cm^3/min. The cup has height 8 cm and radius 6 cm, and is initially full up to the top. Find the rate of change of the height of water in the cup when the cup just begins to leak. Recall the volume of a cone is $V = \pi r^2 h/3$.

5. A hot air balloon on the ground is 200 meters away from an observer. It starts rising vertically at a rate of 50 meters per minute. Find the rate of change of the angle of elevation of the observer when the balloon is 200 meters above the ground.

6. A cannon-ball fired by a cannon at ground level at angle θ to the horizon ($0 \leq \theta \leq \pi/2$) will travel a horizontal distance (called the range, R) given by the formula below:

$$R = \frac{1}{16} v_0^2 \sin \theta \cos \theta.$$

Here $v_0 > 0$, the initial velocity of the cannon-ball, is a fixed constant and air resistance is neglected. What is the maximum possible range?

7. A rectangular trough is 2 meters long, 0.5 meters across the top and 1 meter deep. At what rate must water be poured into the trough such that the depth of the water is increasing at 1 m/min when the depth of the water is 0.7 m?

8. A particular convex lens has a focal length of $f = 10$ cm. The distance p between an object and the lens, the distance q between its image and the lens and the focal length f are related by the equation:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

If an object is 30 cm away from the lens and moving away at 4 cm/sec, how fast is its image moving and in which direction?

9. A pharmaceutical company wants to manufacture cylindrical aluminum cans with a volume of 1000 cm^3 (1 liter). What should the radius and height of the can be to minimize the amount of aluminum used?

10. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

11. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

12. Animals that can produce many healthy babies that survive to the next generation are at an evolutionary advantage over other, com- peting, species. However, too many young produce a heavy burden on the parents (who must feed and care for them). If this causes the parents to die, the advantage is lost. Also, competition of the young with one another for food and parental attention jeopardizes the survival of these babies. Suppose that the evolutionary Advantage A to the parents of having litter size x is

$$A(x) = ax - bx^2.$$

Suppose that the Cost C to the parents of having litter size x is

$$C(x) = mx + e.$$

The Net Reproductive Gain G is defined as $G = A - C$.
(a) At what litter size is the advantage, A, greatest?
(b) At what litter size is there least cost to the parents?
(c) At what litter size is the Net Reproductive Gain greatest?

13. During early development, the limb of a fetus increases in size, but has a constant proportion. Suppose that the limb is roughly a circular cylinder with radius r and length l in proportion $l/r = C$ where C is a positive constant. It is noted that during the initial phase of growth, the radius increases at an approximately constant rate, i.e. that $dr/dt = a$. At what rate does the mass of the limb change during this time? [Note: assume that the density of the limb is 1 gm/cm^3 and recall that the volume of a cylinder is $V = Al$ where A is the base area (in this case of a circle) and l is length.]

Answers, Monday problems (your solution still requires a concluding sentence!)

1. $x = y = 25$.

2. $x = 3$, $y = 2$.

3. (a) $x = y = 10$. (b) $x = y = 10$. (c) $x = 12$, $y = 8$ minimizes $x^3 y^2$.

4. $x = 1/4$ km and $y = 1/2\pi$ km minimize the area of the rectangle.

Answers, Tuesday-Friday problems (your solution still requires a concluding sentence!)

1. Diameter is decreasing at the rate $-1/10\pi$ cm/min when the diameter is 5 cm.

2. Rectangle of maximum area has dimensions $4/\sqrt{3} \times 8/3$.

3. Ladder is sliding at rate $-1/4\sqrt{6}$ m/min when the base is 1m from the wall.

4. Conical cup is leaks at a rate $-1/36\pi$ cm^3/min when it just starts to leak.

5. The change in the angle of elevation is $1/8$ rad/min when the balloon is 200m above the ground.

6. The maximum possible range of the cannonball is $v_0^2/32$.

7. Water must be poured into the trough at rate 1m^3/min so that $dh/dt = 1$m/min when $h = 0.7$m.

8. Assuming the object is moving in a positive direction, the image is moving at the rate of -1cm/sec in the opposite direction from the object (since the image is on the other side of the lens, it's also moving away from the lens).

9. Making the radius $r = 5\sqrt[3]{4/\pi}$ and the height $r = 10\sqrt[3]{4/\pi}$ will minimize the surface area (amount of aluminum required) for a 1 L cylindrical can.

10. The isosceles triangle with maximum area in a circle of radius r has base $\sqrt{3}r$ and height $3r/2$.

11. The point on the curve $y = \sqrt{x}$ with minimum distance to $(3,0)$ is $(5/2, \sqrt{5/2})$.

12. (a) Advantage A is maximized when litter size $x = a/2b$. (b) Cost C is minimized when litter size $x = 0$ (you have to check the endpoints on that one). (c) Net reproductive gain N is maximized by litter size $x = (a - m)/2b$.

13. The mass of the limb increases at the rate $3aC\pi r^2$.