Discontinuous Forcing Functions Example

Problem: Consider an idealized LCR circuit with no resistance such that its natural frequency $1/\sqrt{CL} = \beta$. Assume that initially there is no charge or current in the circuit. From time $t = \pi$ to time $t = 2\pi$ we impose some forcing at some frequency $\omega \neq \beta$ of the form $\sin(\omega t)$. The initial value problem for the charge $q(t)$ is therefore:

$$q'' + \beta^2 q = \begin{cases} 0, & 0 \leq t < \pi \\ \sin(\omega t), & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}, \quad q(0) = q'(0) = 0.$$

What is the charge in the circuit $q(t)$ at time t?

Solution:

We'll solve using Laplace transforms. We begin by writing the RHS using unit step functions.

$$\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right) = \begin{cases} 0, & 0 \leq t < \pi \\ \sin(\omega t), & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}.$$

Our IVP can therefore be re-written as $q'' + \beta^2 q = \sin(\omega t)u_\pi(t) - \sin(\omega t)u_{2\pi}(t)$, $q(0) = q'(0) = 0$. Now we take the Laplace Transform of both sides of the equation. Let $Q(s) = \mathscr{L}\{q(t)\}$. Then

$$\begin{aligned} \mathscr{L}\left\{q'' + \beta^2 q\right\} &= \mathscr{L}\left\{\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)\right\} \\ \mathscr{L}\left\{q''\right\} + \beta^2 \mathscr{L}\{q\} &= \mathscr{L}\left\{\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)\right\}, \end{aligned}$$

since $\mathfrak{L}$ is a linear operator. Going term-by-term:

- $\mathscr{L}\{q''\} = s^2 Q(s) - sq(0) - q'(0) = s^2 Q(s)$ since $q(0) = q'(0) = 0$.
- $\beta^2 \mathscr{L}\{q\} = \beta^2 Q(s)$
- Since we're going to use convolution we won't bother taking the transform of the right hand side - leave as $\mathscr{L}\{\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)\}$

Therefore the transformed IVP $q'' + \beta^2 q = \sin(\omega t)u_\pi(t) - \sin(\omega t)u_{2\pi}(t)$, $q(0) = q'(0) = 0$ is

$$s^2 Q(s) + \beta^2 Q(s) = \mathscr{L}\left\{\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)\right\}$$

Solving for $Q(s)$ we find

$$Q(s) = \frac{1}{s^2 + \beta^2} \mathscr{L}\left\{\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)\right\}$$

To find the charge $q(t)$ all that's left to do is invert the transform, $q(t) = \mathscr{L}^{-1}\{Q(s)\}$. We'll use a convolution integral. Let $F(s) = 1/(s^2 + \beta^2)$, $G(s) = \mathscr{L}\{\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)\}$. Then let $f(t) = \mathscr{L}^{-1}\{F(s)\}$ which we find to be $f(t) = \sin(\beta t)/\beta$ from tables, and $g(t) = \mathscr{L}^{-1}\{G(s)\} = \sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)$. Then from the theorem (6.6.1 in your textbook),

$$Q(s) = F(s)G(s) = \mathscr{L}\left\{\int_0^t f(t-\tau)g(\tau)\,d\tau\right\}$$

with $f(t)$ and $g(t)$ above. Thus since $q(t) = \mathscr{L}^{-1}\{Q(s)\}$,

$$\begin{aligned} q(t) &= \int_0^t f(t-\tau)g(\tau)\,d\tau \\ &= \frac{1}{\beta}\int_0^t \sin\left(\beta(t-\tau)\right)\sin(\omega\tau)\left(u_\pi(\tau) - u_{2\pi}(\tau)\right)\,d\tau. \end{aligned}$$

Note that we could just as easily use $q(t) = \int_0^t f(t)g(t-\tau)\,d\tau$ and obtain the correct result. We choose the above to make our lives easier - $g(t-\tau)$ would give us shifted step functions to integrate.

All that's left to do is integrate! In the following pages we go through the integration quickly and then again more slowly, filling in the details, since integrating over step functions can be tricky.

Fast way:

$$\begin{aligned}
q(t) &= \frac{1}{\beta}\int_0^t \sin(\beta(t-\tau))\sin(\omega\tau)\,(u_\pi(\tau)-u_{2\pi}(\tau))\,d\tau \\
&= \frac{1}{\beta}\int_0^t \sin(\beta(t-\tau))\sin(\omega\tau)u_\pi(\tau)\,d\tau - \frac{1}{\beta}\int_0^t \sin(\beta(t-\tau))\sin(\omega\tau)u_{2\pi}(\tau)\,d\tau \\
&= \left(\frac{1}{\beta}\int_\pi^t \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau\right)u_\pi(t) - \left(\frac{1}{\beta}\int_{2\pi}^t \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau\right)u_{2\pi}(t)
\end{aligned}$$

Then using the trig identity $\sin(\beta(t-\tau))\sin(\omega\tau) = (\cos((\beta+\omega)\tau-\beta t)-\cos((\beta-\omega)\tau-\beta t))/2$

$$\begin{aligned}
q(t) &= \left(\frac{1}{2\beta}\int_\pi^t \cos((\beta+\omega)\tau-\beta t)\,d\tau - \frac{1}{2\beta}\int_\pi^t \cos((\beta-\omega)\tau+\beta t)\,d\tau\right)u_\pi(t) \\
&\quad - \left(\frac{1}{2\beta}\int_{2\pi}^t \cos((\beta+\omega)\tau-\beta t)\,d\tau - \frac{1}{2\beta}\int_{2\pi}^t \cos((\beta-\omega)\tau+\beta t)\,d\tau\right)u_{2\pi}(t) \\
q(t) &= \left\{\frac{\sin(\omega t)}{\beta^2-\omega^2} - \frac{\sin(\pi(\beta+\omega)-\beta t)}{\beta(\beta+\omega)} + \frac{\sin(\pi(\beta-\omega)+\beta t)}{\beta(\beta-\omega)}\right\}u_\pi(t) \\
&\quad + \left\{-\frac{\sin(\omega t)}{\beta^2-\omega^2} + \frac{\sin(2\pi(\beta+\omega)-\beta t)}{\beta(\beta+\omega)} - \frac{\sin(2\pi(\beta-\omega)+\beta t)}{\beta(\beta-\omega)}\right\}u_{2\pi}(t)
\end{aligned}$$

Thus the charge on the circuit at time t is given by the above expression. Plotting the solution we obtain:

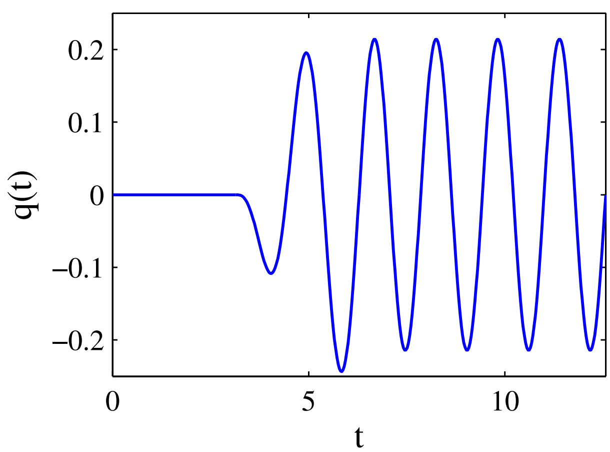

Figure 1: Solution q(t) for $0 \le t \le 4\pi$, for $\beta = 4$ and $\omega = 3$.

Long way, with details:

This task is rendered a *little* trickier in this case b/c of the step functions. Usually a few lines would do it though! (using the trig identity $\sin A \sin B = (\cos(A-B) - \cos(A+B)/2$ to write $\sin(\beta(t-\tau))\sin(\omega\tau) = (\cos((\omega+\beta)\tau - \beta t) - \cos((\omega-\beta)\tau + \beta t))/2)$

However for the purposes of this illustratory example, let's go through the integration steps and deal with the step functions carefully.

Here's the basic idea. Consider the arbitrary function $f(\tau)(u_\pi(\tau) - u_{2\pi}(\tau))$ as depicted in the figure below. Remember the integral gives us the area under the curve. For $t < \pi$ (where $u_\pi(\tau) = 0$ and $u_{2\pi}(\tau) = 0$) notice that the area under the curve is zero! So integrating up to t for $t < \pi$ gives 0. For $\pi \leq t < 2\pi$ the area under the curve from $\tau = 0$ up to $\tau = t$ is the SAME as the area under $f(t)$ from $\tau = \pi$ to $\tau = t$. Finally for $t \geq 2\pi$ the area under the curve is equal to the total area under $f(\tau)$ from $\tau = \pi$ to $\tau = 2\pi$.

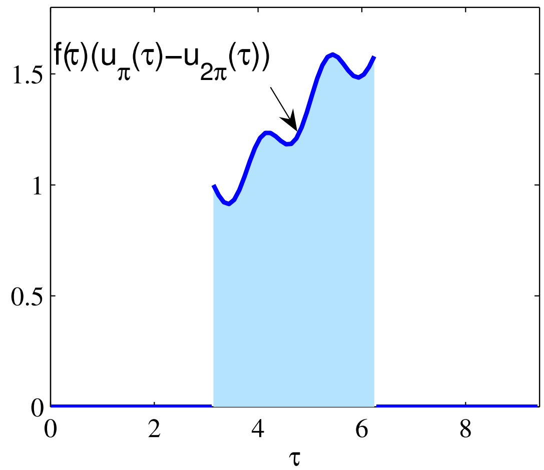

Figure 2: Arbitrary function f(t) (blue, solid line) and the area under the curve (light blue).

Obviously different examples won't necessarily have intervals where the function is zero. But drawing an approximate sketch still sometimes helps us figure out what's going on in those cases, too.

Alternatively you can consider this integral mechanically. What you do is split t up into all the intervals dictated by step functions in the integrand. In this case the "break points" are at $t = \pi$ and $t = 2\pi$ so we consider separately the cases where $0 \leq t < \pi$, $\pi \leq t < 2\pi$, and $t \geq 2\pi$.

Start with $t < \pi$. If this is the case, then the integration runs over $0 \leq \tau \leq t < \pi$. At NO point in this interval is $\tau \geq \pi$ or $\tau \geq 2\pi$, so $(u_\pi(t) - u_{2\pi}(t)) = (0-0) = 0$. Therefore for $t < \pi$,

$$
\begin{aligned}
q(t) &= \frac{1}{\beta}\int_0^t \sin(\beta(t-\tau))\sin(\omega\tau)(u_\pi(\tau) - u_{2\pi}(\tau))\, d\tau \\
&= \frac{1}{\beta}\int_0^t \sin(\beta(t-\tau))\sin(\omega\tau)(0)\, d\tau \\
&= 0.
\end{aligned}
$$

Next up, $\pi \leq t < 2\pi$. Now it's possible for $u_\pi(\tau)$ to be one or zero depending on "where" in the inverval τ finds itself. So let's split the integral up:

$$
\begin{aligned}
q(t) &= \frac{1}{\beta}\int_0^t \sin(\beta(t-\tau))\sin(\omega\tau)(u_\pi(\tau) - u_{2\pi}(\tau))\, d\tau \\
&= \frac{1}{\beta}\int_0^\pi \sin(\beta(t-\tau))\sin(\omega\tau)(u_\pi(\tau) - u_{2\pi}(\tau))\, d\tau + \frac{1}{\beta}\int_\pi^t \sin(\beta(t-\tau))\sin(\omega\tau)(u_\pi(\tau) - u_{2\pi}(\tau))\, d\tau \\
&= \frac{1}{\beta}\int_\pi^t \sin(\beta(t-\tau))\sin(\omega\tau)\, d\tau
\end{aligned}
$$

To get the last line, we note that: (1) the first integral is zero since step functions are zero since throughout the domain of integration and (2) in the second integral, since$\pi \le \tau < 2\pi$ throughout the domain of integration, $u_\pi(\tau) - u_{2\pi}(\tau) = 1 - 0 = 1$. We'll solve it later. Finally, consider $t \ge 2\pi$. Now it's possible for $u_\pi(\tau)$ and/or $u_{2\pi}(t)$ to be one or zero depending on "where" in the inverval τ finds itself. So let's split the integral up:

$$
\begin{aligned}
q(t) &= \frac{1}{\beta}\int_0^t \sin(\beta(t-\tau))\sin(\omega\tau)\left(u_\pi(\tau)-u_{2\pi}(\tau)\right)d\tau \\
&= \frac{1}{\beta}\int_0^\pi \sin(\beta(t-\tau))\sin(\omega\tau)\left(u_\pi(\tau)-u_{2\pi}(\tau)\right)d\tau + \frac{1}{\beta}\int_\pi^{2\pi} \sin(\beta(t-\tau))\sin(\omega\tau)\left(u_\pi(\tau)-u_{2\pi}(\tau)\right)d\tau \\
&+ \frac{1}{\beta}\int_{2\pi}^t \sin(\beta(t-\tau))\sin(\omega\tau)\left(u_\pi(\tau)-u_{2\pi}(\tau)\right)d\tau \\
&= \frac{1}{\beta}\int_\pi^{2\pi} \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau.
\end{aligned}
$$

To get the last line, we note that: (1) the first integral is zero since step functions are zero since throughout the domain of integration, (2) in the second integral, since$\pi \le \tau < 2\pi$ throughout the domain of integration, $u_\pi(\tau) - u_{2\pi}(\tau) = 1 - 0 = 1$, and (3) the last integral is also zero since $\tau \ge 2\pi$ throughout the domain of integration and therefore $u_\pi(\tau) - u_{2\pi}(\tau) = 1 - 1 = 0$. Again we'll solve it later.

In summary we have obtained:

$$
q(t) = \begin{cases} 0, & 0 \le t < \pi \\ \frac{1}{\beta}\int_\pi^t \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau & \pi \le t < 2\pi \\ \frac{1}{\beta}\int_\pi^{2\pi} \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau & t \ge 2\pi \end{cases}
$$

which matches our intuition from sketching. When you're comfortable with integrating over step functions you'll probably be able to just write this down.

To integrate use the trig identity $\sin A \sin B = (\cos(A-B) - \cos(A+B)/2$ to write $\sin(\beta(t-\tau))\sin(\omega\tau) = (\cos((\beta+\omega)\tau - \beta t) - \cos((\beta-\omega)\tau - \beta t))/2$. Then

$$
\begin{aligned}
\int \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau &= \frac{1}{2}\int \cos((\omega+\beta)\tau - \beta t)\,d\tau - \frac{1}{2}\int \cos((\omega-\beta)\tau + \beta t)\,d\tau \\
&= \frac{1}{2(\omega+\beta)}\sin((\omega+\beta)\tau - \beta t) - \frac{1}{2(\omega-\beta)}\sin((\omega-\beta)\tau + \beta t) + C
\end{aligned}
$$

for some arbitrary constant C. Using this solution to compute our definite integrals, we find

$$
\begin{aligned}
\frac{1}{\beta}\int_\pi^t \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau &= \frac{1}{(\beta^2-\omega^2)}\sin(\omega t) - \frac{1}{2\beta(\omega+\beta)}\sin(\pi(\omega+\beta) - \beta t) \\
&+ \frac{1}{2\beta(\omega-\beta)}\sin(\pi(\omega-\beta)\pi + \beta t) \\
\text{and } \frac{1}{\beta}\int_\pi^{2\pi} \sin(\beta(t-\tau))\sin(\omega\tau)\,d\tau &= \frac{1}{2(\omega+\beta)}\left[\sin(2\pi(\omega+\beta) - \beta t) - \sin(\pi(\omega+\beta) - \beta t)\right] \\
&- \frac{1}{2(\omega-\beta)}\left[\sin(2\pi(\omega-\beta) + \beta t) - \sin(\pi(\omega-\beta) + \beta t)\right].
\end{aligned}
$$

Thus we have our solution:

$$
q(t) = \begin{cases} 0, & 0 \le t < \pi \\ \frac{1}{(\beta^2-\omega^2)}\sin(\omega t) - \frac{1}{2\beta(\omega+\beta)}\sin(\pi(\omega+\beta) - \beta t) & \pi \le t < 2\pi \\ \quad + \frac{1}{2\beta(\omega-\beta)}\sin(\pi(\omega-\beta)\pi + \beta t), & \\ \frac{1}{2\beta(\omega+\beta)}\left[\sin(2\pi(\omega+\beta) - \beta t) - \sin(\pi(\omega+\beta) - \beta t)\right] & t \ge 2\pi \\ \quad - \frac{1}{2\beta(\omega-\beta)}\left[\sin(2\pi(\omega-\beta) + \beta t) - \sin(\pi(\omega-\beta) + \beta t)\right], & \end{cases}
$$

is the charge in the circuit at time t. Alternatively, writing this in terms of step functions,

$$q(t) = \left\{ \frac{1}{(\beta^2-\omega^2)}\sin(\omega t) - \frac{\sin(\pi(\omega+\beta)-\beta t)}{2\beta(\omega+\beta)} + \frac{\sin(\pi(\omega-\beta)\pi+\beta t)}{2\beta(\omega-\beta)} \right\} u_{\pi}(t)$$
$$+ \left\{ -\frac{1}{(\beta^2-\omega^2)}\sin(\omega t) + \frac{\sin(2\pi(\omega+\beta)-\beta t)}{2\beta(\omega+\beta)} - \frac{\sin(2\pi(\omega-\beta)+\beta t)}{2\beta(\omega-\beta)} \right\} u_{2\pi}(t).$$

Using trig identities you can show this is the same as the solution we obtained using partial fractions. The solution is plotted below.

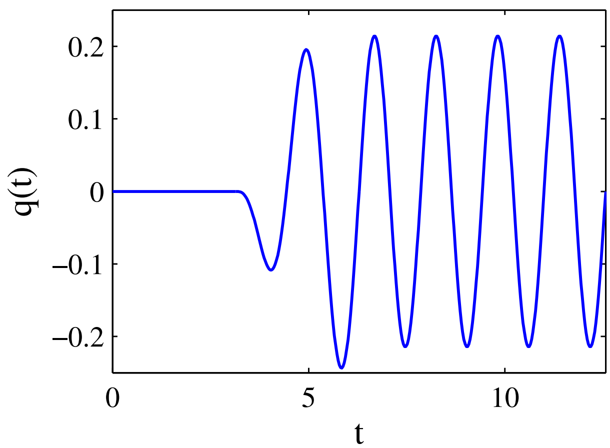

Figure 3: Solution q(t) for $0 \le t \le 4\pi$, for $\beta = 4$ and $\omega = 3$.