Computing Eigenvalues and Eigenvectors

Example 1:

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}.$$

Solution:

The eigenvalues λ and eigenvectors $\vec{v}$ satisfy the matrix equation $A\vec{v} = \lambda\vec{v}$ or, re-arranging terms,

$$(A - \lambda I)\vec{v} = 0, \text{ where } I \text{ is the } 2 \times 2 \text{ identity matrix } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order for $(A - \lambda I)\vec{v} = 0$ to have a non-trivial (i.e., non-zero) solution, a necessary and suffucient condition is:

$$\det(A - \lambda I) = 0.$$

For an $n \times n$ matrix A, this will give us an $n^{th}-$order polynomial, the characteristic equation, whose roots are the eigenvalues λ. In this case $n = 2$. Finding this characteristic equation,

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left\{ \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} &= 0 \\ \det \begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} &= 0 \\ (2-\lambda)(-2-\lambda)+3 &= 0 \\ \lambda^2 - 1 &= 0. \end{aligned}$$

Therefore our characteristic equation is $\lambda^2 - 1 = 0$. Its roots, $\lambda = \pm 1$, are the eigenvalues. Now we need to find corresponding eigenvectors.

To find the eigenvector $\vec{v_1}$ corresponding to the eigenvalue $\lambda_1 = 1$ we must solve $(A - \lambda_1 I)\vec{v_1} = 0$. Substituting for A and λ_1 gives

$$\begin{aligned} \left[\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where $\vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Note that this matrix equation is equivalent to the single scalar equation $u_1 - 3u_2 = 0$. Therefore solutions are found setting an arbitrary value for one of u_1 or u_2 - say, $u_2 = s$ - and determining the other of u_1 or u_2 accordingly - here, $u_1 = 3s$. Thus the eigenvectors associated with the eigenvalue $\lambda_1 = 1$ can be written as

$$\vec{v_1} = s \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

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Aside: Notice that eigenvectors are unique but only up to a multiplicative factor!

Showing this: Assume $\vec{v}$ is the eigenvector associated with the eigenvalue λ of the matrix A. We can show that $s\vec{v}$ is also an eigenvector associated with the eigenvalue λ of the matrix A for any scalar s: We know that λ, $\vec{v}$ satisfy the equation $A\vec{v} = \lambda\vec{v}$ (by the definition of eigenvalues of eigenvectors/eigenvalues). Multiplying both sides by the scalar s we obtain $sA\vec{v} = s\lambda\vec{v}$. Exploiting commutativity of scalar multiplication, we can write this as $A(s\vec{v}) = \lambda(s\vec{v})$. Thus $s\vec{v}$ is also an eigenvector associated with the eigenvalue λ of the matrix A.

Note: ALL eigenvectors associated with the eigenvalue $\lambda_1 = 1$, when joined with the zero vector, form a subspace. The same is true for $\lambda_2 = -1$. These subspaces are called *eigenspaces*.

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To find the eigenvector $\vec{v}_2$ corresponding to the eigenvalue $\lambda_2 = -1$ we must solve $(A - \lambda_2 I)\vec{v}_2 = 0$. Substituting for A and λ_2 gives

$$\begin{aligned}\left[\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},\end{aligned}$$

where $\vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.Note that this matrix equation is equivalent to the single scalar equation $u_1 - u_2 = 0$. Therefore solutions are found setting an arbitrary value for one of u_1 or u_2 - say, $u_2 = s$ - and determining the other of u_1 or u_2 accordingly - here, $u_1 = s$. Thus the eigenvectors associated with the eigenvalue $\lambda_2 = -1$ can be written as

$$\vec{v}_2 = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Answer: The matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding eigenvectors for $\lambda_1 = 1$ are $\vec{v}_1 = s\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and the corresponding eigenvectors for $\lambda_2 = -1$ are $\vec{v}_2 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, for arbitrary scalar parameters s and t.

Example 2: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution: The eigenvalues λ and eigenvectors $\vec{v}$ satisfy the matrix equation $A\vec{v} = \lambda\vec{v}$ or, re-arranging terms,

$$(A - \lambda I)\vec{v} = 0, \text{ where } I \text{ is the } 2 \times 2 \text{ identity matrix } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order for $(A - \lambda I)\vec{v} = 0$ to have a non-trivial (i.e., non-zero) solution, a necessary and suffucient condition is:

$$\det(A - \lambda I) = 0.$$

For an $n \times n$ matrix A, this will give us an $n^{th}-$order polynomial, the characteristic equation, whose roots are the eigenvalues λ. In this case $n = 2$. Finding this characteristic equation,

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left\{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} &= 0 \\
\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} &= 0 \\
(1-\lambda)^2 + 1 &= 0 \\
(\lambda - 1)^2 &= -1 \\
\lambda - 1 &= \pm i \\
\lambda &= 1 \pm i
\end{aligned}$$

Therefore our characteristic equation is $(\lambda - 1)^2 + 1 = 0$. Its roots, $\lambda = 1 \pm i$, are the eigenvalues. Now we need to find corresponding eigenvectors.

To find an eigenvector $\vec{v_1}$ corresponding to the eigenvalue $\lambda_1 = 1 + i$ we must solve $(A - \lambda_1 I)\vec{v_1} = 0$. Substituting for A and λ_1 gives

$$\begin{aligned}
\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - (1+i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -iu_1 + u_2 \\ -u_1 - iu_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

where $\vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Note that this matrix equation is equivalent to the single scalar equation $-iu_1 + u_2 = 0$.

How is that equivalent to the other scalar equation $-u_1 - iu_2 = 0$? Multiply both sides of the equation by i to get

$$(i)(-u_1) - i(iu_2) = 0 \Rightarrow -iu_1 + u_2 = 0.$$

Therefore solutions are found setting an arbitrary value for one of u_1 or u_2 - say, $u_2 = 1$ - and determining the other of u_1 or u_2 accordingly - here, $u_1 = -i$. Thus an eigenvector associated with the eigenvalue $\lambda_1 = 1 + i$ is

$$\vec{v_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

To find an eigenvector $\vec{v_2}$ corresponding to the eigenvalue $\lambda_2 = 1 - i$ we must solve $(A - \lambda_2 I)\vec{v_2} = 0$. Substituting for A and λ_2 gives

$$\begin{aligned}
\left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} - (1-i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} iu_1 + u_2 \\ -u_1 + iu_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

where $\vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.Note that this matrix equation is equivalent to the single scalar equation $iu_1 + u_2 = 0$. Therefore solutions are found setting an arbitrary value for one of u_1 or u_2 - say, $u_2 = 1$ - and determining the other of u_1 or u_2 accordingly - here, $u_1 = i$. Thus an eigenvectors associated with the eigenvalue $\lambda_2 = 1 - i$ is

$$\vec{v_2} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Answer: The matrix A has eigenvalue $\lambda_1 = 1 + i$ with corresponding eigenvector $\vec{v_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ and eigenvalue $\lambda_2 = 1 - i$ with corresponding eigenvector $\vec{v_2} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

But remember any scalar multiple of the eigenvectors is still an eigenvector! For example, $\vec{v_1} = \begin{pmatrix} 1 \\ i \end{pmatrix}$, too.

Example 3:

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}.$$

Solution:

The eigenvalues λ and eigenvectors $\vec{v}$ satisfy the matrix equation $A\vec{v} = \lambda\vec{v}$ or, re-arranging terms,

$$(A - \lambda I)\vec{v} = 0, \text{ where } I \text{ is the } 3 \times 3 \text{ identity matrix } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In order for $(A - \lambda I)\vec{v} = 0$ to have a non-trivial (i.e., non-zero) solution, a necessary and suffucient condition is:

$$\det(A - \lambda I) = 0.$$

For an $n \times n$ matrix A, this will give us an $n^{th}-$order polynomial, the characteristic equation, whose roots are the eigenvalues λ. In this case $n = 2$. Finding this characteristic equation,

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left\{ \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} &= 0 \\ \det \begin{pmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)[-\lambda(5-\lambda)+4] - 2[(5-\lambda)-4] - 1[-4+4\lambda] &= 0 \\ (\lambda-1)(\lambda-2)(\lambda-3) &= 0, \end{aligned}$$

after some simplification. The roots of this characteristic equation correspond to the eigenvalues of A. Thus we find that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.

Now we need to find the eigenvectors. To find the eigenvector $\vec{v_1}$ corresponding to the eigenvalue $\lambda_1 = 1$ we must solve $(A - \lambda_1 I)\vec{v_1} = 0$. Substituting for A and λ_1 gives

$$\begin{aligned}
\left[\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - (1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

where $\vec{v_1} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Using Gaussian elimination (elementary row operations), we can find that this matrix equation is equivalent to the scalar equations (1) $2u_2 - u_3 = 0$ and (2) $u_1 - u_2 + u_3 = 0$. We can obtain solutions by setting an aribtrary value to one of the u_1, u_2, or u_3 - say $u_2 = s$. Then from (1) we find $u_3 = 2s$; then after that from (2) we find $u_1 = -s$. Thus the eigenvectors associated with the eigenvalue $\lambda_1 = 1$ can be written as

$$\vec{v_1} = s\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v_1} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ (see note in previous problem concerning uniqueness of eigenvectors only up to multiplicative constants).

To find the eigenvector $\vec{v_2}$ corresponding to the eigenvalue $\lambda_2 = 2$ we must solve $(A - \lambda_2 I)\vec{v_2} = 0$. Substituting for A and λ_2 gives

$$\begin{aligned}
\left[\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - (2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

where $\vec{v_2} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Using Gaussian elimination (elementary row operations), we can find that this matrix equation is equivalent to the scalar equations (1) $u_1 - 2u_2 + u_3 = 0$ and (2) $4u_1 - 4u_2 + 3u_3 = 0$. We can obtain solutions by setting an aribtrary value to one of the u_1, u_2, or u_3 - say $u_2 = s$. Then from solving (1) we find $u_3 = 2s - u_1$; plugging that into (2) we find $4u_1 - 4s + 3(2s - u_1) = 0$ or $u_1 = -2s$. Then $u_3 = 4s$. Thus the eigenvectors associated with the eigenvalue $\lambda_2 = 2$ can be written as

$$\vec{v_2} = s\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$.

Finally, to find the eigenvector $\vec{v}_3$ corresponding to the eigenvalue $\lambda_3 = 3$ we must solve $(A - \lambda_3 I)\vec{v}_3 = 0$. Substituting for A and λ_3 gives

$$\left[\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - (3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\vec{v}_3 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Using Gaussian elimination (elementary row operations), we can find that this matrix equation is equivalent to the scalar equations (1) $-2u_1 + 2u_2 - u_3 = 0$ and (2) $u_1 - 3u_2 + u_3 = 0$. We can obtain solutions by setting an aribtrary value to one of the u_1, u_2, or u_3 - say $u_1 = s$. Then from solving (1) we find $u_3 = -2s + 2u_2$; plugging that into (2) we find $s - 3u_2 + (-2s + 2u_2) = 0$ or $u_2 = -s$. Then $u_3 = -4s$. Thus the eigenvectors associated with the eigenvalue $\lambda_3 = 2$ can be written as

$$\vec{v}_3 = s\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}.$$

If we want AN eigenvector, we can choose $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$.

Answer: The matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The corresponding eigenvectors for $\lambda_1 = 1$ are $\vec{v}_1 = r\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, the corresponding eigenvectors for $\lambda_2 = 2$ are $\vec{v}_2 = s\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, and the corresponding eigenvectors for $\lambda_3 = 2$ are $\vec{v}_3 = t\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$, for arbitrary scalar parameters r, s and t.