Applications Examples - First Order Equations

Problems and Solutions

Problem 1: The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-hour cycle, then the rate of change in the level of hormone in the blood may be represented by the initial value problem:

$$\frac{dx}{dt} = \alpha - \beta \cos\left(\frac{\pi t}{12}\right) - kx, \qquad x(0) = x_0,$$

where $x(t)$ is the amount of hormone in the blood at time t, α is the average secretion rate, β is the amount of daily variation in the secretion, and k is a positive constant reflecting the rate at which the body removes hormones from the blood. If $\alpha = \beta = 1$, $k = 2$, and $x_0 = 10$, solve for $x(t)$.

Problem 2: The temperature T (in units of 100°F) of a university classroom on a cold winter day varies with time t (in hours) as:

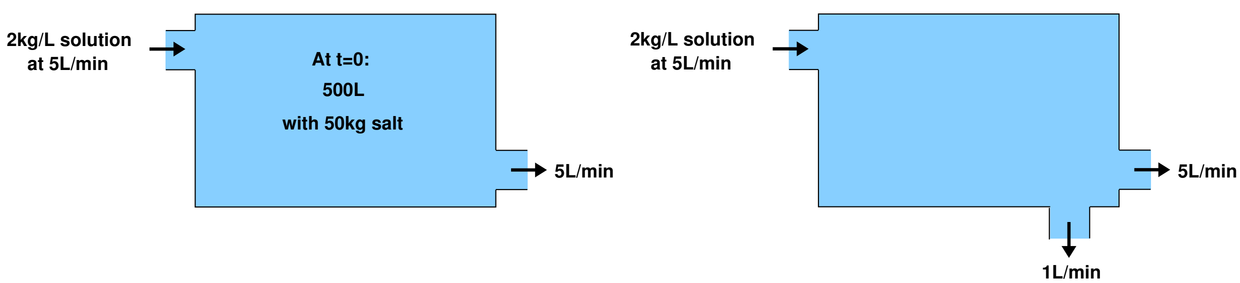
$$\frac{dT}{dt} = \begin{cases} 1 - T, & \text{if the heating unit is ON.} \\ -T, & \text{if the heating unit is OFF.} \end{cases}$$

Suppose that $T = 0$ at 9:00am, the heating unit is ON from 9-10am, OFF from 10-11am, ON again from 11am-noon, and so on for the rest of the day.

(a) How warm will the classroom be at noon? At 5pm?

(b) Plot the solution for $0 < t < 72$ hours.

Problem 3: Suppose a brine containing 2kg of salt per liter runs into a tank initially filled with 500L of water containing 50kg of salt. The brine enters the tank at a rate of 5 L/min. The mixture, kept uniform by stirring, is flowing out at a rate of 5 L/min. See figure (a).



(a) $0 < t \leq 10$min (b) $t > 10$min

(a) Find the concentration, in kg/L of salt in the tank after 10 min.

Hint: Let A denote the number of kg of salt in the tank at t minutes after the process begins and use the fact that:

rate of increase of A = rate of input - rate of exit.

(b) After 10 min, a leak develops in the tank and an additional liter per minute of mixture flows out of the tank (see figure (b)). What will be the concentration, in kg/L of salt in the tank, 20 min after the leak develops?

Problem 1 Solution:
First we re-write the differential equation representing hormone level $x(t)$ in standard form: $\frac{dx}{dt} + kx = \alpha - \beta \cos\left(\frac{\pi t}{12}\right) - kx$. α, β, and k are given constants and we could immediately plug in their values, but instead we'll do that at the end. We can solve this differential equation using an integrating factor: $\mu(t) = e^{\int^t k\, ds} = e^{kt}$. Using the integrating factor, we obtain:

$$\frac{d}{dt}\left(e^{kt}x\right) = \alpha e^{kt} - \beta e^{kt}\cos\left(\frac{\pi t}{12}\right) \Rightarrow x(t) = e^{-kt}\int^t (\alpha e^{ks} - \beta e^{ks}\cos\left(\frac{\pi s}{12}\right))\, ds$$

Integrating by parts we obtain:

$$x(t) = \frac{\alpha}{k} - \beta\left(\frac{k}{k^2 + (\pi/12)^2}\cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12(k^2 + (\pi/12)^2)}\sin\left(\frac{\pi t}{12}\right)\right) + Ce^{-kt}.$$

C is an unknown constant, found by applying the initial condition: $x(0) = x_0$ so $C = x_0 - \alpha/k + \beta k/(k^2 + (\pi/12)^2)$. The solution to the initial value problem is:

$$x(t) = \frac{\alpha}{k} - \beta\left(\frac{k}{k^2 + (\pi/12)^2}\cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12(k^2 + (\pi/12)^2)}\sin\left(\frac{\pi t}{12}\right)\right) + e^{-kt}\left(x_0 - \frac{\alpha}{k} + \frac{\beta k}{(k^2 + (\pi/12)^2)}\right).$$

We are given $\alpha = \beta = 1$, $k = 2$, and $x_0 = 10$; plugging in those values, we say the hormone level in the blood is given by:

$$x(t) = \frac{1}{2} - \left(\frac{2}{4 + (\pi/12)^2}\cos\left(\frac{\pi t}{12}\right) + \frac{\pi}{12(4 + (\pi/12)^2)}\sin\left(\frac{\pi t}{12}\right)\right) + e^{-2t}\left(10 - \frac{1}{2} + \frac{2}{(4 + (\pi/12)^2)}\right).$$

Problem 2 Solution:
We'll solve this problem step-by-step. When the heat is turned ON, the temperature T is given by:

$$\frac{dT}{dt} + T = 1, \quad T(t_0) = T_0$$

where time $t = t_0$ corresponds to the time the heat is turned ON (i.e. 9am, 11am, 1pm,...) and T_0 is the temperature of the room at that time. Note that we've put the equation in standard form. Using the integrating factor $\mu(t) = e^t$ and applying the initial condition, we find that the temperature when the heat is turned ON is:

$$T(t) = 1 + (T_0 - 1)e^{-(t-t_0)}.$$

When the heat is turned OFF, the temperature T is given by:

$$\frac{dT}{dt} + T = 0, \quad T(t_0) = T_0$$

where time $t = 0$ corresponds to the time the heat is turned OFF (i.e. 10am, 12pm, 2pm,...) and T_0 is the temperature of the room at that time. Using the integrating factor $\mu(t) = exp(t)$ and applying the initial condition, we find that the temperature when the heat is turned OFF is:

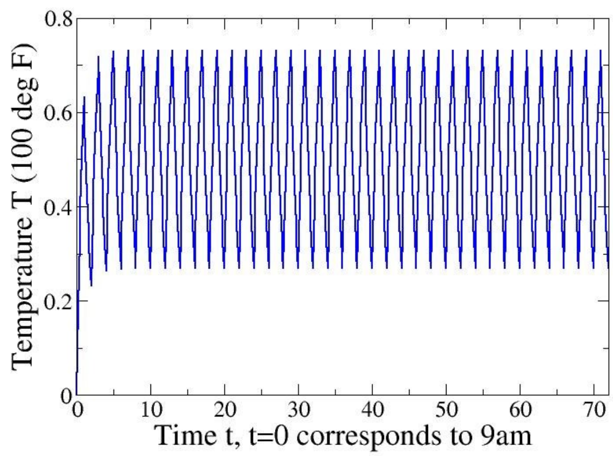
$$T(t) = T_0 e^{-(t-t_0)}.$$

Now let's start looking at what happens through the day, setting $t = 0$ as the time t corresponding to 9am. At $t = 0$ (at 9am) the temperature is $T_0 = 0°F$ and the heat is turned ON. The temperature is therefore given by: $T(t) = 1 - e^{-t}$, as $t_0 = 0$. At 10am ($t = 1$), after 1 hour, the temperature is $1 - e^{-1}$. The heat is turned OFF, so the temperature is now given by $T(t) = (1 - e^{-1})e^{-(t-1)}$, as now $t_0 = 1$, and $T(t_0) = T_0 = 1 - e^{-1}$, the temperature when the heat is turned OFF. At 11am ($t = 2$), after 1 hour, the temperature is $(1 - e^{-1})e^{-1}$. The heat is turned ON, so the temperature is now given by $T(t) = 1 + ((1 - e^{-1})e^{-1} - 1)e^{-(t-2)}$, as now

$t_0 = 2$, and $T(t_0) = T_0 = (1 - e^{-1})e^{-1}$, the temperature when the heat is turned ON. Proceeding in this fashion we notice that at the integer times t_{ON} when the heat is turned ON, and integer times t_{OFF} when the heat is turned OFF, the temperature is:

$$T(t_{ON}) = \sum_{n=1}^{t_{ON}} (-1)^{n+1} e^{-n} \quad \text{and} \quad T(t_{OFF}) = \sum_{n=0}^{t_{OFF}} (-1)^n e^{-n}.$$

Then at noon ($t = 3$), when the heat is turned OFF, the temperature is $T(3) = 1 - e^{-1} + e^{-2} - e^{-3} \approx 0.72$. At 5pm ($t = 8$), when the heat is turned ON, the temperature is $T(8) = e^{-1} - e^{-2} + e^{-3} - e^{-4} + e^{-5} - e^{-6} + e^{-7} - e^{-8} \approx 0.27$. You may have found these by iterating, without noticing the pattern, and that's fine, too.
(b) Since the heat is being turned on and off, the temperature fluctuates, which can be readily seen in a plot of the temperature:



Notice that, after a short initial transient, the temperature fluctuations are regular.

Problem 3 Solution:
(a) Let's consider the tank as a container of salt, and let $A(t)$ denote the amount of salt in the tank at time t. We can recover the concentration by dividing $s(t)$ by the volume of fluid in the tank. Initially, there are 50kg of salt in the tank. The differential equation describing the amount of salt in the tank at a time t is:

$$\frac{dA}{dt} = \text{rate of salt entering} - \text{rate of salt exiting}, \quad A(0) = 50$$

The rate at which salt ENTERS the tank is: (rate brine flows into tank)×(concentration of salt in the brine)=(5L/min)×(2kg/L)=10kg/min.

The rate at which salt EXITS the tank is (rate brine flows out of tank)×(concentration of salt in the tank at time t)=5L/min×(concentration of salt in the tank at time t). To find the concentration of salt in the tank, note that the tank at all times is well-stirred so the concentration in the tank is uniform. Since brine is entering and exiting the tank at the same rate, the volume of fluid in the tank is a constant 500L. The concentration is then $A(t)$kg/500L and so the rate at which salt EXITS the tank=(5L/min)×($A(t)$kg/500L)=($A(t)$/100)kg/min.

We thus have an initial value problem for the amount of salt $A(t)$:

$$\frac{dA}{dt} = 10 - \frac{A}{100}, \quad A(0) = 50.$$

Solving using the integrating factor $\mu(t) = e^{t/100}$, we find the general solution $A(t) = 1000 + Ce^{-t/100}$. By applying the initial condition $A(0) = 50$ we find $C = -950$ and that the amount of salt $A(t)$ at time t is:

$$A(t) = 1000 - 950e^{-t/100}.$$

Then after 10 minutes the amount of salt in the tank is $A(10) = 1000 - 950e^{-1/10} \approx 140.4$kg, and the concentration of salt in the tank is $(1000 - 950e^{-1/10})$kg/500L ≈ 0.28kg/L.

(b) With the additional leak the volume V of the tank is now changing:

$$\text{rate of change of volume} = (\text{volume in})/\text{time} - (\text{volume out})/\text{time}.$$

Since the initial volume in the tank V(0)=500L, we can write an initial value problem for the volume in the tank with respect to time t:

$$\frac{dV}{dt} = 5\text{L/min} - (5\text{L/min} + 1\text{L/min}) = -1\text{L/min} \qquad V(10) = 500\text{L}.$$

Solving this, $V(t) = -t + C$ and, applying the initial condition, $C = 510$ and the volume in the tank is $V(t) = 510 - t$.

The equation for the amount of salt is dealt with in a manner similar to part (a):

$$\frac{dA}{dt} = \text{rate of salt entering} - \text{rate of salt exiting}, \qquad A(0) = 50$$

The rate of amount of salt ENTERING is still 10kg/min. And rate of amount of salt EXITING the tank is still equal to the (rate at which brine flows out of tank)$\times$(concentration of salt in the tank at time t). But now the rate out is 5L/min+1L/min=6L/min, and the concentration of salt in the tank at time t is $A(t)/V(t) = A(t)/(510 - t)$, from above. The rate of amount of salt EXITING the tank is then $(6\text{L/min}) \times (A(t)/(510 - t)\,\text{kg/L}) = 6A(t)/(510 - t)\,\text{kg/min}$.

We thus have an initial value problem for the amount of salt $A(t)$:

$$\frac{dA}{dt} = 10 - \frac{6A}{510 - t}, \qquad A(10) = 1000 - 950e^{-1/10}.$$

The initial condition $A(10) = 1000 - 950e^{-1/10}$ is from part (a). Solving using the integrating factor $\mu(t) = 1/(510 - t)^6$, we find the general solution $A(t) = 2(510 - t) + C(510 - t)^6$. By applying the initial condition we find $C = -950e^{-1/10}/500^6$ and that the amount of salt $A(t)$ at time t is:

$$A(t) = 2(510 - t) - 950e^{-1/10}\left(\frac{510 - t}{500}\right)^6$$

. Then 20 minutes after the leak develops - corresponding to $t = 30$ - the amount of salt in the tank is $A(30) = 2(510 - 30) - 950e^{-1/10}((510 - 30)/500)^6 \approx 287.14$kg. The volume of salt at $t = 30$ is $V(30) = 510 - 30 = 480$L. Then the concentration of salt in the tank is at $t = 30$, 20 minutes after the leak develops, is 287.14kg/480L ≈ 0.60kg/L.