Undetermined Vectors Example

Problem: Find the general solution of the linear system of equations

$$\begin{aligned}\frac{dx}{dt} &= -2x - y + 9t \\ \frac{dy}{dt} &= x - 4y + 2e^t\end{aligned}$$

Solution:

The system of equations is linear, it can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 9t \\ 2e^t \end{pmatrix}.$$

Let $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$.

The homogeneous solution $\vec{x}_h(t)$ is

$$\vec{x}_h(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + C_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t}\right]$$

(we already computed this; see notes on repeated eigenvalues).

To find the particular solution $\vec{x}_p$ we'll use the method of undetermined vectors. First we write down our equation in such a way that we identify constant vector coefficients to the t-dependency in the inhomogeneity. That is,

$$\begin{pmatrix} t \\ 2e^t \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^t.$$

Then we seek to find the particular solution $\vec{x}_p$ such that $\vec{x}_p$ satisfies

$$\vec{x}_p' = A\vec{x}_p + \begin{pmatrix} 9 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^t$$

using the method of undetermined vectors.

This is just like before, with second order linear equations! For just a second, think of the inhomogeneity as $K_1 t + K_2 e^t$ (where K_1 and K_2 are some known constants). If we had $ay'' + by' + cy = K_1 t + K_2 e^t$ and we wanted to find the particular solution y_p using the method of undetermined coefficients, we would use the trial expression $y_p = At + B + Ce^t$.

Here, instead of undetermined coefficients, we have undetermined vectors - that's the only difference. Choose the trial expression

$$\vec{x}_p = \vec{a}t + \vec{b} + \vec{c}e^t$$

where $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ are our "unknown vectors." Now all that's left to do is plug into the equation and compute what $\vec{a}$, $\vec{b}$, $\vec{c}$ must be.

$$\begin{aligned}\vec{x}_p' &= A\vec{x}_p + \begin{pmatrix} 9 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^t \\ \frac{d}{dt}\left[\vec{a}t + \vec{b} + \vec{c}e^t\right] &= A\left[\vec{a}t + \vec{b} + \vec{c}e^t\right] + \begin{pmatrix} 9 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^t \\ \vec{a} + \vec{c}e^t &= (A\vec{a})\,t + \left(A\vec{b}\right) + (A\vec{c})\,e^t + \begin{pmatrix} 9 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^t.\end{aligned}$$

We move everything over to the left hand side and find the coefficients of t^0, t^1, e^t:

$$0 = \left(A\vec{b} - \vec{a}\right) + \left(A\vec{a} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) t + \left(A\vec{c} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \vec{c}\right) e^t.$$

Thus we have 3 matrix equations for our 3 unknown vectors:

$$A\vec{b} - \vec{a} = 0 \quad (1)$$

$$A\vec{a} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad (2)$$

$$A\vec{c} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \vec{c} = 0. \quad (3)$$

Let's solve (2) first. $A\vec{a} + \begin{pmatrix} 9 \\ 0 \end{pmatrix} = 0 \Rightarrow A\vec{a} = \begin{pmatrix} -9 \\ 0 \end{pmatrix} \Rightarrow \vec{a} = A^{-1}\begin{pmatrix} -9 \\ 0 \end{pmatrix}$. Since $A^{-1} = \frac{1}{9}\begin{pmatrix} -4 & 1 \\ -1 & -2 \end{pmatrix}$,

$$\vec{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Next let's solve (1). $A\vec{b} - \vec{a} = 0 \Rightarrow \vec{b} = A^{-1}\vec{a}$. Since $A^{-1} = \frac{1}{9}\begin{pmatrix} -4 & 1 \\ -1 & -2 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$,

$$\vec{b} = \begin{pmatrix} -15/9 \\ -6/9 \end{pmatrix} = \begin{pmatrix} -5/3 \\ -2/3 \end{pmatrix}.$$

Finally, let's solve (3). $A\vec{c} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \vec{c} = 0 \Rightarrow (A - I)\vec{c} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, where I is the 2×2 identity matrix. Then $\vec{c} = (A-I)^{-1}\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. Since $A - I = \begin{pmatrix} -3 & -1 \\ 1 & -5 \end{pmatrix}$, $(A-I)^{-1} = \frac{1}{16}\begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix}$ and so

$$\vec{c} = \begin{pmatrix} -2/16 \\ 6/16 \end{pmatrix} = \begin{pmatrix} -1/8 \\ 3/8 \end{pmatrix}.$$

Thus our particular solution is

$$\vec{x}_p = \begin{pmatrix} 4 \\ 1 \end{pmatrix} t - \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} -1/8 \\ 3/8 \end{pmatrix} e^t.$$

Finally we can write down the general solution to the given linear system of ODEs,

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h + \vec{x}_p \\ &= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + C_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} \right] + \begin{pmatrix} 4 \\ 1 \end{pmatrix} t - \begin{pmatrix} 5/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} -1/8 \\ 3/8 \end{pmatrix} e^t. \end{aligned}$$

With an initial condition $\vec{x}(0)$ the problem is an IVP; we can determine C_1 and C_2 using the initial condition to solve the IVP.