Dec. 2009

1. (a) $-y' + y = 4e^{3x}$

Standard form: $y' - y = -4e^{3x}$

Compute IF : $\mu(x) = e^{-\int dx} = e^{-x}$

Multiply by $\mu(x)$, put in integrable form: $e^{-x}y' - e^{-x}y = -4e^{2x}$

$$\frac{d}{dx}(e^{-x}y) = -4e^{2x}$$

Integrate: $e^{-x}y = -2e^{2x} + C$

$$y = -2e^{3x} + Ce^{x}$$

Therefore the general solution is $\boxed{y(x) = -2e^{3x} + Ce^{x}}$

(b) $y' = 4x(y-1)^{1/2}$, $y > 1$.

Eq is separable: $\int \frac{dy}{2(y-1)^{1/2}} = \int 2x\,dx$

$$(y-1)^{1/2} = x^2 + C$$

$$y - 1 = (x^2+C)^2$$

$$y = (x^2+C)^2 + 1.$$

Therefore the general solution is $\boxed{y(x) = (x^2+C)^2 + 1}$

#2 - NOT course material this term.

#3. Use the method of variation of parameters to find the general solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

$$y'' + 2y' + y = 3e^{-t}. \quad \leftarrow \underline{\text{NOTE}}\text{: ALREADY in standard } \underline{\text{form}}.$$

First find homog. sol., $y_h(t)$ s.t. $y_h'' + 2y_h' + y_h = 0$.

Let $y = e^{rt}$, obtain char. eq. $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$, repeated root.

Therefore the homogeneous solutions are

$$y_1(t) = e^{-t}, \quad y_2(t) = te^{-t}.$$

Now find particular solution using variation of parameters.

Let $y_p = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$.

Then $u_1(t), u_2(t)$ must satisfy $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$ where $g(t) = 3e^{-t}$.

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W[y_1, y_2]}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

So $u_1(t) = \int \frac{-y_2(t)\,g(t)}{W[y_1, y_2]}\,dt$; $u_2(t) = \int \frac{y_1(t)\,g(t)}{W[y_1, y_2]}\,dt$

Now, $W[y_1, y_2] = y_1 y_2' - y_2 y_1' = (e^{-t})(e^{-t} - te^{-t}) - (te^{-t})(-e^{-t})$

$= e^{-2t}$

So $u_1(t) = \int -\frac{(te^{-t})(3e^{-t})}{(e^{-2t})}\,dt = -3\int t\,dt = -\frac{3t^2}{2}$.

$u_2(t) = \int \frac{(e^{-t})(3e^{-t})}{e^{-2t}}\,dt = 3\int dt = 3t$.

and $y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$

$$= \left(\frac{-3t^2}{2}\right)(e^{-t}) + (3t)(te^{-t}) = \frac{-3t^2e^{-t}}{2} + 3t^2e^{-t}$$

$$\boxed{y_p(t) = \frac{3}{2}t^2e^{-t}}$$

Check particular solution using method of undetermined coefficients.
Since $3e^{-t}$ is our inhomogeneity ($g(t)$ above), but e^{-t} and te^{-t} are homogeneous solutions, our trial guess should be:

$$y_p = At^2e^{-t}.$$

then $y_p' = 2Ate^{-t} - At^2e^{-t}$, $y_p'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$.

plug into ODE : $y_p'' + 2y_p' + y_p = 3e^{-t}$.

$$(2Ae^{-t} - 4Ate^{-t} + \cancel{At^2e^{-t}}) + 2(2Ate^{-t} - \cancel{At^2e^{-t}}) + (\cancel{At^2e^{-t}}) = 3e^{-t}$$

$$2Ae^{-t} = 3e^{-t}$$

$$(2A - 3)e^{-t} = 0$$

so $2A - 3 = 0$ or $A = 3/2$.

Thus $\boxed{y_p(t) = \frac{3}{2}t^2e^{-t}}$

All together, the general solution is $y(t) = y_h(t) + y_p(t)$ or

$$\boxed{y(t) = C_1e^{-t} + C_2te^{-t} + \frac{3}{2}t^2e^{-t}}$$

4. Force balance – assume down is positive.

$$F = mg - kx - cv \quad \text{or} \quad ma = mg - kx - cv$$

but $a = \frac{d^2x}{dt^2}$, $v = \frac{dx}{dt}$, so this is $mx'' + cx' + kx = mg$.

NOTE: using "x" instead of "u"; distance from unstretched position.

Given $m = 1g$, $c = 20\, g/s$, $k = ?$

Obtain k from force balance at rest: $kx = mg$

$$k(5cm) = (1g)(980\, cm/s^2)$$

$$k = 196\, g/s^2.$$

Ok and the initial condition is $x(0) = \underset{\text{at rest}}{5cm} + \underset{\text{stretch}}{2cm} = 7cm$. ; $x'(0) = 0$ ("let go")

Thus, our IVP is:

$$\begin{cases} x'' + 20x' + 196x = 980 \\ x(0) = 7, \quad x'(0) = 0 \end{cases}$$

Start by finding homogeneous solution

$$x_h'' + 20x_h' + 196x_h = 0$$

Let $x_h = e^{rt}$, recover characteristic eq. $r^2 + 20r + 196 = 0$.

with roots $r = \frac{-20 \pm \sqrt{(20)^2 - 4(1)(196)}}{2} = -10 \pm \frac{\sqrt{384}\,i}{2}$

or $r = -10 \pm 4\sqrt{6}\,i$ (since $\sqrt{384} = 8\sqrt{6}$)

So the homogeneous solution is $x_h(t) = C_1 e^{-10t}\cos(4\sqrt{6}t) + C_2 e^{-10t}\sin(4\sqrt{6}t)$.

Now find particular sol, using the method of undet coefs.

Trial guess: $x_p(t) = A$; $x_p' = x_p'' = 0$, Plugging in, $196A = 980$

$$A = 5.$$

Thus the general solution is

$$x(t) = C_1 e^{-10t}\sin(4\sqrt{6}t) + C_2 e^{-10t}\cos(4\sqrt{6}t) + 5$$

(notice $\lim_{t\to\infty} x(t) = 5cm$, spring returns to rest position)

Finally use ICs to compute C_1, C_2.

$$X(0) = C_1 + 5 = 7 \Rightarrow C_1 = 2.$$

$$X'(0) = \dots = -10C_1 + 4\sqrt{6}\,C_2 = 0 \Rightarrow C_2 = \frac{5\sqrt{6}}{6}.$$

Therefore the spring's position x at time t is given by

$$x(t) = 2e^{-10t}\cos(4\sqrt{6}t) + \frac{5\sqrt{6}}{6}e^{-10t}\sin(4\sqrt{6}t) + 5.$$

(distance from unstretched spring length)

5.(a) Solve the IVP $\begin{cases} y'' + y = u_{\pi/2}(t) + \delta(t - \pi/6) \\ y(0) = y'(0) = 0 \end{cases}$

Use Laplace transforms. Let $\mathcal{L}\{y\} = Y(s)$.

Then $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

$\mathcal{L}\{u_{\pi/2}\} = e^{-\frac{\pi}{2}s}/s.$

$\mathcal{L}\{\delta(t-\pi/6)\} = e^{-\frac{\pi}{6}s}.$

Our transformed equation is therefore.

$$s^2Y(s) + Y(s) = e^{-\pi/2 s}/s + e^{-\pi/6 s}$$

Solving for $Y(s)$: $Y(s) = \dfrac{e^{-\pi/2 s}}{s(s^2+1)} + \dfrac{e^{-\pi/6 s}}{s^2+1}$

$$= \frac{e^{-\pi/2 s}}{s} - \frac{s e^{-\frac{\pi}{2}s}}{s^2+1} + \frac{e^{-\frac{\pi}{6}s}}{s^2+1}. \qquad \left(\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.\right)$$

Now we take the inverse transform to recover $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$\mathcal{L}^{-1}\left\{\frac{e^{-\frac{\pi}{2}s}}{s}\right\} = u_{\pi/2}(t).$$

$$\mathcal{L}^{-1}\left\{\frac{s e^{-\frac{\pi}{2}s}}{s^2+1}\right\} = \cos(t - \tfrac{\pi}{2})u_{\pi/2}(t)$$ since $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u_c(t)$ and $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.$

$$= \sin t\, u_{\pi/2}(t).$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi/6\, s}}{s^2+1}\right\} = \sin(t-\pi/6)\,u_{\pi/6}(t)$$ since $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)\,u_c(t)$, $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$$= \left(\frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t\right)u_{\pi/6}(t).$$

Therefore, $\boxed{y(t) = \left(\frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t\right)u_{\pi/6}(t) + \left(1-\sin t\right)u_{\pi/2}(t)}$

(b) Solve the IVP $\begin{cases} y'' + 4y = \frac{1}{k}\left[(t-5)\,u_5(t) - (t-(5+k))\,u_{5+k}(t)\right] \\ y(0) = y'(0) = 0 \end{cases}$

Use Laplace transforms. Let $\mathcal{L}\{y\} = Y(s)$.

Then going term-by-term:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$$\mathcal{L}\left\{\tfrac{1}{k}(t-5)u_5(t)\right\} = \frac{e^{-5s}}{ks^2}$$ since $\mathcal{L}\{f(t-c)u_c(t)\} = e^{-cs}F(s)$ where $F(s) = \mathcal{L}\{f(t)\}$ ($f(t) = t$ here).

$$\mathcal{L}\left\{\tfrac{1}{k}(t-(5+k))u_{5+k}(t)\right\} = \frac{e^{-(5+k)s}}{ks^2}.$$

Then the transformed ODE is:

$$s^2Y(s) + 4Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Solving for $Y(s)$: $$Y(s) = \frac{e^{-5s}}{ks^2(s^2+4)} - \frac{e^{-(5+k)s}}{ks^2(s^2+4)}$$

Now $$\frac{1}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s^2+4}$$ (partial fractions decomposition.

$$= \frac{(A+B)s^2 + 4A}{s^2(s^2+4)}$$

$$\Rightarrow A = 1/4,\quad B = -1/4$$

So $$Y(s) = \frac{1}{4k}\cdot\frac{e^{-5s}}{s^2} - \frac{1}{4k}\cdot\frac{e^{-5s}}{s^2+4} - \frac{1}{4k}\cdot\frac{e^{-(5+k)s}}{s^2} + \frac{1}{4k}\cdot\frac{e^{-(5+k)s}}{s^2+4}.$$

Now take the inverse transform to obtain $y(t)$ $\left(y(t) = \mathcal{L}^{-1}\{Y(s)\}\right)$.

$$\mathcal{L}^{-1}\{e^{-5s}/s^2\} = (t-5)u_5(t)$$

$$\mathcal{L}^{-1}\left\{e^{-5s}/s^2+4\right\} = \sin[2(t-5)]\,u_5(t)$$

(as $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u_c(t)$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \sin 2t$.

$$\mathcal{L}^{-1}\left\{\frac{e^{-(5+k)}}{s^2}\right\} = [t-(5+k)]\,u_{5+k}(t).$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-(5+k)}}{s^2+4}\right\} = \sin[2(t-(5+k))]\,u_{5+k}(t).$$

Putting it together:

$$y(t) = \frac{1}{4k}\left\{\left[t-5 \mp \sin(2(t-5))\right]u_5(t) - \left[t-(5+k) - \sin(2(t-(5+k)))\right]u_{5+k}(t)\right\}$$

is the solution to the IVP.

Note that at $k \to 0$, ~~they~~ get $y(t)=0$! Which makes sense:
$y(t)=0$ is the solution to $\begin{cases} y''+4y=0 \\ y(0)=y'(0)=0. \end{cases}$

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6. For the linear system

$$\frac{d}{dt}\vec{x} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}\vec{x}$$

find the general solution and plot a few trajectories.

Let $A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$

The eigenvalues of A satisfy $\det(A-rI)=0 \Rightarrow \det\begin{pmatrix} -r & 4 \\ 1 & -r \end{pmatrix} = 0$

or $r^2-4=0$

$\therefore$ the eigenvalues are $r_1 = 2$, $r_2 = -2$.

(real, opposite sign $\to$ unstable saddle node).

The corresponding eigenvectors $\vec{\xi}_{1,2}$ satisfy $(A - r_{1,2} I)\vec{\xi}_{1,2} = \vec{0}$.

For $r_1 = 2$: $(A - 2I)\vec{\xi}_1 = \vec{0}$ or $\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \vec{\xi}_1 = \vec{0}$ so $\vec{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

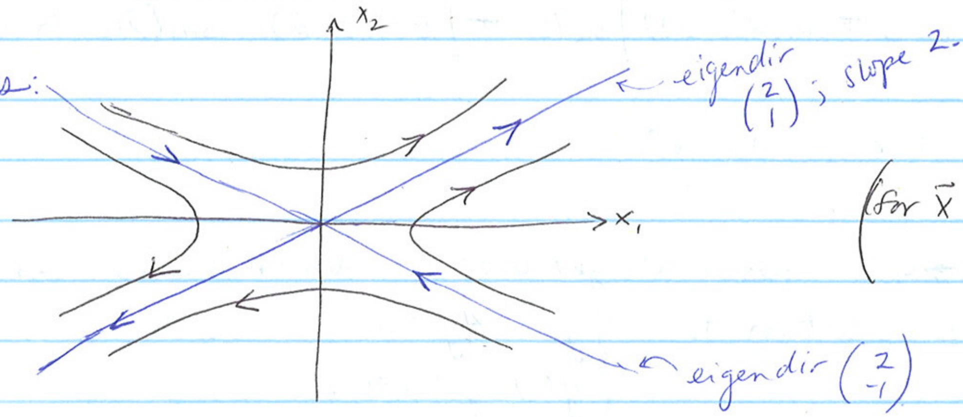
$r_2 = -2$: $(A + 2I)\vec{\xi}_2 = \vec{0}$ or $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \vec{\xi}_2 = \vec{0}$ so $\vec{\xi}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Therefore the general solution is

$$\vec{x} = c_1 \vec{\xi}_1 e^{r_1 t} + c_2 \vec{\xi}_2 e^{r_2 t}$$

or $$\boxed{\vec{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-2t}}$$

Trajectories:



(for $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$)

7. Find the general solution of the following system

$$\frac{d}{dt}\vec{x} = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{=A} \vec{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}.$$

First we find the homogeneous solution, $\vec{x}_h$ s.t. $\vec{x}_h' = A\vec{x}_h$.

Find eigenvalues and eigenvectors of A.

Eigenvalues, r, of A satisfy $\det(A - rI) = 0 \Rightarrow \det\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} = 0$

$\Rightarrow (1-r)^2 - 4 = 0 \Rightarrow r_1 = 3,\ r_2 = -1$ are the e-vals.

Corresponding eigenvectors $\vec{\xi}_{1,2}$ satisfy $(A - r_{1,2} I)\vec{\xi}_{1,2} = \vec{0}$.

For $r_1 = 3$: $(A - 3I)\vec{\xi}_1 = \vec{0} \Rightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\vec{\xi}_1 = \vec{0} \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $r_2 = -1$: $(A + I)\vec{\xi}_2 = \vec{0} \Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\vec{\xi}_2 = \vec{0} \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Therefore the homogeneous solution is $\vec{x}_h = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$.

Now find the particular solution. Use undetermined vectors.

Trial guess: $\vec{x}_p = \vec{a}\, e^t$; $\vec{x}_p{}' = \vec{a}\, e^t$.

Plug into ODE: $\vec{x}_p{}' = \cancel{A\vec{a}}\ A\vec{x}_p + \vec{g}\, e^t$ (let $\vec{g} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$)

& solve for $\vec{a}$ | $(\vec{a}\, e^t) = A(\vec{a}\, e^t) + \vec{g}\, e^t$.

$$(A\vec{a} - \vec{a} + \vec{g})e^t = 0.$$

but $e^t \neq 0$ & so $A\vec{a} - \vec{a} + \vec{g} = 0$

$$(A - I)\vec{a} = -\vec{g}$$

$$\vec{a} = -(A - I)^{-1}\vec{g}$$

Now since $A - I = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$, $(A - I)^{-1} = \frac{1}{-4}\begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1/4 \\ +1 & 0 \end{pmatrix}$

and $\vec{a} = -\begin{pmatrix} 0 & +1/4 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix}$

Therefore $\vec{x}_p = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix}e^t$.

And the general solution is $\vec{x}(t) = \vec{x}_h + \vec{x}_p$

or $\boxed{\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t} + \begin{pmatrix} 1/4 \\ -2 \end{pmatrix}e^{-t}}$

8. Consider the system $\begin{cases} x'(t) = x(y-1) & = F(x,y) \\ y'(t) = 2+x-y & = G(x,y). \end{cases}$ $(*)$

(a) Find the Jacobian.

$$J(x,y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \quad \text{or} \quad \boxed{J(x,y) = \begin{pmatrix} y-1 & x \\ 1 & -1 \end{pmatrix}}$$

(b) Find the critical points for the system $(*)$.

Critical points when $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

$\frac{dx}{dt} = 0 \Rightarrow x(y-1) = 0$ so $x = 0$ and/or $y = 1$.

$\frac{dy}{dt} = 0 \Rightarrow 2+x-y = 0$. If $x = 0$, $2-y = 0$ so $y = 2$.

$y = 1$, $2+x-1 = 0$ so $x = -1$.

Therefore the critical points are $(0,2)$ and $(-1,1)$.

(c) Use linear systems near critical points to classify them (type/stability).

Near $(x,y) = (0,2)$:

Let $x = 0 + \tilde{x}$

$y = 2 + \tilde{y}$ for $\tilde{x}, \tilde{y}$ small.

Then $\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = J(0,2)\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ or $\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$.

$J(0,2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ has eigenvalues r_1, r_2 s.t. $\det(J(0,2) - rI) = 0$

or $(1-r)(-1-r) = 0$

$\Rightarrow r_1 = 1, r_2 = -1$.

eigenvalues are real, with opposite sign.

$\therefore$ $(0,2)$ is an unstable, saddle point.

Near $(x,y) = (-1,1)$

Let $x = -1 + \tilde{x}$

$y = 1 + \tilde{y}$.

Then $\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = J(-1,1)\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$.

$J(-1,1) = \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$ has e-vals r that satisfy

$\det(J(-1,1) - rI) = 0$

or $-r(-1-r) + 1 = 0$

$r^2 + r + 1 = 0$

so $r = \dfrac{-1 \pm \sqrt{(1) - 4(1)(1)}}{2}$

$\rightsquigarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

eigenvalues are complex, with -ve real part.

$\therefore$ $(-1,1)$ is a stable, spiral point.

$\boxed{\begin{array}{l}\text{In summary: critical point } (0,2) \text{ is an unstable saddle node.}\\ \text{critical point } (-1,1) \text{ is a stable spiral point}\end{array}}$