Method of Partial Fractions Expansion

From:

Nagle, R.K., Saff, E.B., and Snider, A.D., 2000. Fundamentals of Differential Equations and Boundary Value Problems. New York: Addison-Wesley.

Pages 392-396.

We briefly review this method. Recall from calculus that a rational function of the form $P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials with the degree of P less than the degree of Q, has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. (We assume the coefficients of the polynomials to be real numbers.) There are three cases to consider:

1. Nonrepeated linear factors.
2. Repeated linear factors.
3. Quadratic factors.

1. Nonrepeated Linear Factors

If $Q(s)$ can be factored into a product of distinct linear factors,

$$Q(s) = (s - r_1)(s - r_2) \cdots (s - r_n) ,$$

where the r_i's are all distinct real numbers, then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \cdots + \frac{A_n}{s - r_n} ,$$

where the A_i's are real numbers. There are various ways of determining the constants $A_1, \ldots, A_n$. In the next example, we demonstrate two such methods.

EXAMPLE 5 Determine $\mathscr{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s - 1}{(s + 1)(s + 2)(s - 3)} .$$

SOLUTION We begin by finding the partial fraction expansion for $F(s)$. The denominator consists of three distinct linear factors, so the expansion has the form

$$\frac{7s - 1}{(s + 1)(s + 2)(s - 3)} = \frac{A}{s + 1} + \frac{B}{s + 2} + \frac{C}{s - 3} , \qquad (6)$$

where A, B, and C are real numbers to be determined.

One procedure that works for all partial fraction expansions is first to multiply the expansion equation by the denominator of the given rational function. This leaves us with two identical polynomials. Equating the coefficients of s^k leads to a system of linear equations that we can solve to determine the unknown constants. In this example, we multiply (6) by $(s+1)(s+2)(s-3)$ and find

(7) $$7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)\ ,^{\dagger}$$

which reduces to

$$7s - 1 = (A + B + C)s^2 + (-A - 2B + 3C)s + (-6A - 3B + 2C).$$

Equating the coefficients of s^2, s, and 1 gives the system of linear equations

$$\begin{aligned} A + B + C &= 0\ , \\ -A - 2B + 3C &= 7\ , \\ -6A - 3B + 2C &= -1\ . \end{aligned}$$

Solving this system yields $A = 2$, $B = -3$, and $C = 1$. Hence

(8) $$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\ .$$

An alternative method for finding the constants A, B, and C from (7) is to choose three values for s and substitute them into (7) to obtain three linear equations in the three unknowns. If we are careful in our choice of the values for s, the system is easy to solve. In this case, equation (7) obviously simplifies if $s = -1, -2$, or 3. Putting $s = -1$ gives

$$\begin{aligned} -7 - 1 &= A(1)(-4) + B(0) + C(0)\ , \\ -8 &= -4A\ . \end{aligned}$$

Hence $A = 2$. Next, setting $s = -2$ gives

$$\begin{aligned} -14 - 1 &= A(0) + B(-1)(-5) + C(0)\ , \\ -15 &= 5B\ , \end{aligned}$$

and so $B = -3$. Finally, letting $s = 3$, we similarly find that $C = 1$. In the case of nonrepeated linear factors, the alternative method is easier to use.

Now that we have obtained the partial fraction expansion (8), we use linearity to compute

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\right\}(t) \\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &\quad + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t) \\ &= 2e^{-t} - 3e^{-2t} + e^{3t}\ . \end{aligned}$$ ■ End

† Rigorously speaking, equation (7) was derived for s different from -1, -2, and 3, but by continuity it holds for these values as well.

2. Repeated Linear Factors

Let $s - r$ be a factor of $Q(s)$ and suppose $(s - r)^m$ is the highest power of $s - r$ that divides $Q(s)$. Then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s - r)^m$ is

$$\frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \cdots + \frac{A_m}{(s-r)^m} ,$$

where the A_i's are real numbers.

EXAMPLE 6 Determine $\mathscr{L}^{-1}\left\{\dfrac{s^2 + 9s + 2}{(s-1)^2(s+3)}\right\}$.

SOLUTION Since $s - 1$ is a repeated linear factor with multiplicity two and $s + 3$ is a nonrepeated linear factor, the partial fraction expansion has the form

$$\frac{s^2 + 9s + 2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3} .$$

We begin by multiplying both sides by $(s - 1)^2(s + 3)$ to obtain

(9) $$s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2 .$$

Now observe that when we set $s = 1$ (or $s = -3$), two terms on the right-hand side of (9) vanish, leaving a linear equation that we can solve for B (or C). Setting $s = 1$ in (9) gives

$$1 + 9 + 2 = A(0) + 4B + C(0) ,$$
$$12 = 4B ,$$

and hence $B = 3$. Similarly, setting $s = -3$ in (9) gives

$$9 - 27 + 2 = A(0) + B(0) + 16C$$
$$-16 = 16C .$$

Thus $C = -1$. Finally, to find A, we pick a different value for s, say $s = 0$. Then, since $B = 3$ and $C = -1$, plugging $s = 0$ into (9) yields

$$2 = -3A + 3B + C = -3A + 9 - 1$$

so that $A = 2$. Hence

(10) $$\frac{s^2 + 9s + 2}{(s-1)^2(s+3)} = \frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3} .$$

We could also have determined the constants A, B, and C by first rewriting equation (9) in the form

$$s^2 + 9s + 2 = (A + C)s^2 + (2A + B - 2C)s + (-3A + 3B + C) .$$

Then, equating the corresponding coefficients of s^2, s, and 1 and solving the resulting system, we again find $A = 2$, $B = 3$, and $C = -1$.

Now that we have derived the partial fraction expansion (10) for the given rational function, we can determine its inverse Laplace transform:

$$\mathscr{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t) = \mathscr{L}^{-1}\left\{\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}\right\}(t)$$

$$= 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t)$$

$$- \mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}(t)$$

$$= 2e^t + 3te^t - e^{-3t} \,.$$ ■ End

3. Quadratic Factors

Let $(s-\alpha)^2+\beta^2$ be a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients. Suppose m is the highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$. Then the portion of the partial fraction expansion that corresponds to $(s-\alpha)^2+\beta^2$ is

$$\frac{C_1s+D_1}{(s-\alpha)^2+\beta^2}+\frac{C_2s+D_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{C_ms+D_m}{[(s-\alpha)^2+\beta^2]^m}\,.$$

As we saw in Example 4, it is more convenient to express C_is+D_i in the form $A_i(s-\alpha)+\beta B_i$ when we look up the Laplace transforms. So let's agree to write this portion of the partial fraction expansion in the equivalent form

$$\frac{A_1(s-\alpha)+\beta B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2(s-\alpha)+\beta B_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{A_m(s-\alpha)+\beta B_m}{[(s-\alpha)^2+\beta^2]^m}\,.$$

EXAMPLE 7 Determine $\mathscr{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

SOLUTION We first observe that the quadratic factor s^2-2s+5 is irreducible (check the sign of the discriminant in the quadratic formula). Next we write the quadratic in the form $(s-\alpha)^2+\beta^2$ by completing the square:

$$s^2-2s+5=(s-1)^2+2^2\,.$$

Since s^2-2s+5 and $s+1$ are nonrepeated factors, the partial fraction expansion has the form

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)}=\frac{A(s-1)+2B}{(s-1)^2+2^2}+\frac{C}{s+1}\,.$$

When we multiply both sides by the common denominator, we obtain

(11) $$2s^2+10s=[A(s-1)+2B](s+1)+C(s^2-2s+5)\,.$$

In equation (11), let's put $s = -1$, 1, and 0. With $s = -1$, we find

$$\begin{aligned} 2 - 10 &= [A(-2) + 2B](0) + C(8) , \\ -8 &= 8C , \end{aligned}$$

and hence $C = -1$. With $s = 1$ in (11), we obtain

$$2 + 10 = [A(0) + 2B](2) + C(4) ,$$

and since $C = -1$, the last equation becomes $12 = 4B - 4$. Thus $B = 4$. Finally, setting $s = 0$ in (11) and using $C = -1$ and $B = 4$ gives

$$\begin{aligned} 0 &= [A(-1) + 2B](1) + C(5) , \\ 0 &= -A + 8 - 5 , \\ A &= 3 . \end{aligned}$$

Hence $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1} .$$

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1}\right\}(t) \\ &= 3\mathscr{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 2^2}\right\}(t) \\ &\quad + 4\mathscr{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 2^2}\right\}(t) - \mathscr{L}^{-1}\left\{\frac{1}{s + 1}\right\}(t) \\ &= 3e^t\cos 2t + 4e^t\sin 2t - e^{-t} . \end{aligned}$$

■ End