1. Find the solution $y(t)$ to

$$y' = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} -3 \\ 3 \end{pmatrix}.$$

Also, describe the behavior of the solution for large t.

Solution:
The characteristic polynomial is

$$\lambda^2 + 5\lambda + 4 = 0$$

with roots -1, -4. The eigenvector equation for -1 is $-x + y = 0$, and that for -4 is $x + 2y = 0$. The solutions are therefore

$$y = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

For the initial conditions given we solve

$$c_1 + 2c_2 = -3$$
$$c_1 - c_2 = 3\,,$$

with solutions $c_1 = 1$, $c_2 = -2$. All solutions decay exponentially to 0.

2. Find the solution $y(t)$ to

$$y' = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Solution:
The characteristic polynomial is

$$\lambda^2 - 4\lambda + 4 = 0$$

so $\lambda = 2$. This is the exceptional case. The eigenvector equation is $-2x + y = 0$, leading to an eigenvector $\xi = (1, 2)$. We solve

$$(A - 2I)\eta = \xi$$

to get an equation $-2x + y = 1$ for η, or $\eta = (0, 1)$. Therefore the solutions are

$$y = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \left[t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

For the initial conditions we solve

$$c_1 = 1$$
$$2c_1 + c_2 = 3\,,$$

leading to $c_1 = 1$, $c_2 = 1$. All solutions grow exponentially fast as $t \to \infty$.

3. Find the solution $y(t)$ to

$$y' = \begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix} y, \quad \text{with} \quad y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solution:

The characteristic polynomial

$$\lambda^2 + 6\lambda + 13 = 0$$

leading to roots $-3 \pm 2i$. Eigenvector equation $-ix + y = 0$. Therefore the real solutions are the real and imaginary parts of

$$e^{(-3+2i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-3t} \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + ie^{-3t} \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

The solution is therefore,

$$y = c_1 e^{-3t} \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

The initial conditions give $c_1 = 1$, $c_2 = 2$. Solutions oscillate and decay.