- Review problems posted on website
- Special Office Hours Fri 2:30-4:30
- Midterm: Lecture material up to and including Wed Sept. 22.
- Email on Sat: math255s104.fall2010@gmail.com

Eg Solve the IVP

$$\begin{cases} y'' + \frac{1}{4}y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$$

Let $y = e^{rt}$ so $y' = re^{rt}$, $y'' = r^2e^{rt}$.

Plug in: $y'' + \frac{1}{4}y = 0$

$$r^2e^{rt} + \frac{1}{4}e^{rt} = 0$$

since $e^{rt} \neq 0$, $\div e^{rt}$ to obtain

$$\boxed{r^2 + \frac{1}{4} = 0}$$ characteristic equation

roots: $r = \pm i \cdot \frac{1}{2}$.

therefore $e^{\frac{i}{2}t}$, $e^{-\frac{i}{2}t}$ are solutions.

as we showed last time, equivalent to saying $\sin(t/2)$, $\cos(t/2)$ are solutions.

<u>General solution</u>

$$y(t) = C_1 \sin\left(\frac{t}{2}\right) + C_2 \cos\left(\frac{t}{2}\right)$$

Solve for C_1 & C_2:

$$y(0) = 1 = C_1 \sin\left(\frac{0}{2}\right) + C_2 \cos\left(\frac{0}{2}\right)$$

$$= C_2$$

so $C_2 = 1$.

$$y'(0) = 0 = \frac{1}{2} C_1 \cos\left(\frac{t}{2}\right) - \frac{1}{2} C_2 \sin\left(\frac{t}{2}\right)\Big|_{t=0}$$

$$= \frac{1}{2} C_1$$

so $C_1 = 0$

and $\boxed{y(t) = \cos(t/2)}$ is the solution to the IVP.

Working on

$$ay'' + by' + cy = 0$$

char. eq.

$$ar^2 + br + c = 0.$$

roots $\quad r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Cases ...

Case I : $b^2 - 4ac < 0$. (Mon)

Case II : $b^2 - 4ac > 0$. (Last Wed).

$\boxed{\text{Case III}}$ $\quad b^2 - 4ac = 0$

Then roots

$$r_{1,2} = \frac{-b}{2a} \pm \cancelto{0}{\frac{\sqrt{b^2 - 4ac}}{2a}}$$

$$= \frac{-b}{2a}.$$

$r = -b/2a$ is a REPEATED ROOT.

So we have 1 solution to ODE

$$y_1(t) = e^{\frac{-b}{2a}t}.$$

What's the general solution?

Need a second, linearly indep. solution.

Use <u>Reduction of Order</u>.

Reduction of order method:

For any 2^{nd} order linear DE.

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Suppose we know one solution $y_1(t)$.

$$L[y_1] = 0.$$

To find a second solution,

Let $y_2(t) = v(t) \cdot y_1(t)$

and solve for $v(t)$.

<u>Plug in</u>:

$$y_2' = v' \cdot y_1 + v y_1'$$

$$y_2'' = v'' \cdot y_1 + 2v' y_1' + v y_1''$$

$$L[y_2] = y_2'' + p(t) y_2' + q(t) y_2 = 0$$

$$= (v'' \cdot y_1 + 2v' y_1' + \cancel{v y_1''})$$

$$+ p(t)(v' \cdot y_1 + \cancel{v \cdot y_1'})$$

$$+ \cancel{q(t) \cdot v \cdot y_1} = 0$$

$$L[y_2] = \underbrace{\{y_1'' + p\cdot y_1' + q\cdot y_1\}}_{} v + v'\{2y_1' + p\cdot y_1\} + v''\{y_1\} = 0$$

$= 0$. We know $L[y_1] = 0$.

Left with.

$$y_1(t)\cdot\frac{d^2v}{dt^2} + \left(2\frac{dy_1}{dt} + p(t)\,y_1\right)\frac{dv}{dt} = 0. \quad (*)$$

First order differential eq. for $\dfrac{dv}{dt}$

(order "reduced").

Let $w = \dfrac{dv}{dt} \Rightarrow y_1\dfrac{dw}{dt} + (2y_1' + p\cdot y_1)w = 0$

- Solve $(*)$ for $\dfrac{dv}{dt}$
- Integrate to obtain $v(t)$.
- Gives us our second solution,

$$y_2(t) = v(t)\cdot y_1(t).$$

In the present case,

$$ay'' + by' + cy = 0$$

where $b^2 - 4ac = 0$

$y_1(t) = e^{-\frac{b}{2a}t}$ is one solution.

Use reduction of order to get the other.

ODE: $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0.$

Let $y_2(t) = v(t) \cdot y_1(t) = v(t)e^{-b/2a\, t}$

be the second solution, solve for v.

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

Plug in.

$$0 = y_2'' + \frac{b}{a}y_2' + \frac{c}{a}y_2$$

$$= (v''y_1 + 2v'y_1' + vy_1'')$$

$$+ \frac{b}{a}(v'y_1 + vy_1') + \frac{c}{a}vy_1.$$

$$= v\underbrace{\{y_1'' + \frac{b}{a}y_1' + \frac{c}{a}y_1\}}_{=0}$$

$$+ v'\{\frac{b}{a}y_1 + 2y_1'\} + v'' \cdot y_1$$

$$0 = v'' y_1 + v' \left\{ \frac{b}{a} y_1 + 2 y_1' \right\}$$

But $y_1 = e^{-\frac{b}{2a} t}$ so $y_1' = \frac{-b}{2a} e^{-\frac{b}{2a} t}$

and $0 = v'' \cdot e^{-\frac{b}{2a} t} + v' \left\{ \frac{b}{a} e^{-\frac{b}{2a} t} + 2\left(\frac{-b}{2a} e^{-\frac{b}{2a} t} \right) \right\}$

$$0 = e^{-\frac{b}{2a} t} \left\{ v'' + v' \left[\frac{b}{a} \cancel{e^{-\frac{b}{2a} t}} - \frac{b}{a} \right] \right\}$$

$$0 = e^{-\frac{b}{2a} t} (v'')$$

but $e^{-\frac{b}{2a} t} \neq 0 \Rightarrow \boxed{v'' = 0}$

$$v'' = 0$$
$$v' = c_1$$
$$v = c_1 t + c_2.$$

Only interested in t-dependence (constants w/in general sol).

$$\therefore \quad y_2(t) = v(t) \cdot y_1(t)$$
$$= t \cdot e^{-b/2a\, t}.$$

is our second solution.

Case III $b^2 - 4ac = 0$.

2 linearly independent solutions are

$$y_1(t) = e^{r_1 t}$$
$$y_2(t) = t e^{r_1 t} \quad \left(r_1 = \frac{-b}{2a}\right)$$

General solution is

$$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}.$$

To prove y_1, y_2 linearly indep compute W.

$$W[y_1, y_2] = y_1' y_2 - y_1 y_2'$$
$$= (r_1 e^{r_1 t}) t e^{r_1 t} - (e^{r_1 t})(\cancel{t} e^{r_1 t} + r_1 t e^{r_1 t})$$
$$\neq 0.$$