LECTURE 11

Announcements:

- Recall midterm MONDAY.
- Extra office hours today, 2:30-4.
  (MATX 1110 or 1101)
- HW 4 posted (due Fri. Oct. 8):
  p. 163 # 2, 17, 29, 32.
  p. 171 # 23
  p. 183 # 17, 28.
- Correction to nomenclature.
  - $e^{i\theta} = [\cancel{\sin\theta + i\cos}]\ \cos\theta + i\sin\theta$ → "Euler's Formula"
  - $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$
    → "de Moivre's formula* (or theorem)"
    (can show using Euler's formula)

(sorry for any confusion).

We had $ay'' + by' + cy = 0$ (**)

with associated characteristic eq.

$$ar^2 + br + c = 0$$

that has roots $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

We were considering

Case III - $b^2 - 4ac = 0$ → "Repeated root"

Found 2 linearly independent solutions to ODE (**)

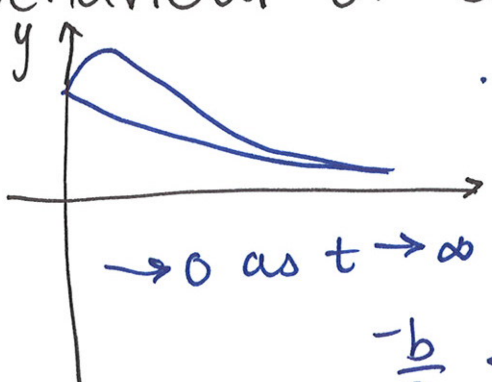
$$y_1(t) = e^{r_1 t}, \quad y_2(t) = t e^{r_1 t}$$

where $r_1 = -b/2a$.

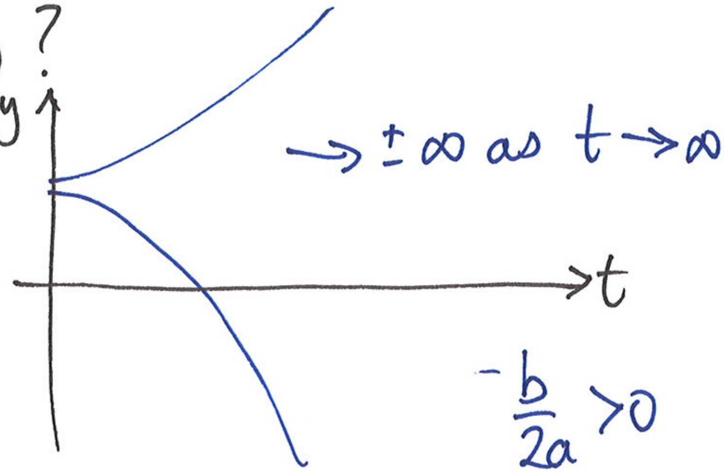
The general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}, \quad r_1 = -b/2a.$$

Behaviour of solution?



$\frac{-b}{2a} < 0$



$\frac{-b}{2a} > 0$

Recall mass-spring system.

$$mx'' + bx' + kx = 0$$

mass, damping, spring constant.

"$-\frac{b}{2a}$" $= \frac{-b}{2m}$ (in this case)

$b>0,\ m>0 \rightsquigarrow r_1 = \frac{-b}{2m} < 0.$

Left plot is behaviour.

"$b^2 - 4ac$" $\Longrightarrow$ $b^2 - 4mk = 0$

$b^2 = 4mk.$

damping.

"critically damped case"

<u>Eg</u> Find the general solution of:

$$y'' + 2y' + y = 0.$$

Let $y = e^{rt}$ then $y' = re^{rt}$, $y'' = r^2 e^{rt}$.

Plug in; ODE becomes

$$r^2 e^{rt} + 2re^{rt} + e^{rt} = 0$$

$$(r^2 + 2r + 1)e^{rt} = 0$$

$e^{rt} \neq 0 \Longrightarrow r^2 + 2r + 1 = 0$

char. eq.

<u>Find roots</u>: $(r+1)^2 = 0$

so $r = -1$ is a repeated root.

Then e^{-t}, te^{-t} are solutions to the ODE

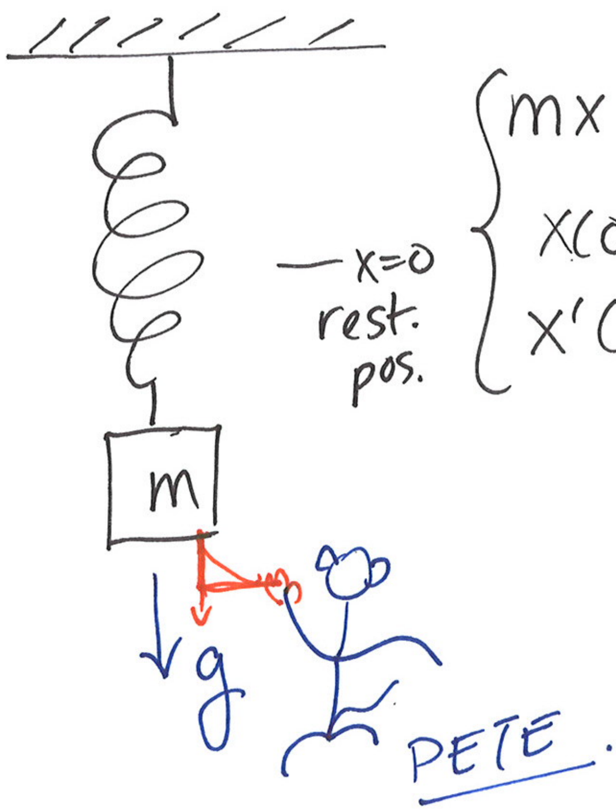
And the general solution is

$$\underline{y(t) = c_1 e^{-t} + c_2 e^{-t}\, t \;\Big|}$$

Non-homogeneous equations

(section 3.6 of text; more later).

What if we forced our oscillator?



$$\begin{cases} mx'' + bx' + kx = mg + f(t) \\ x(0) = \text{initial pos} \\ x'(0) = \text{initial vel.} \end{cases}$$

<u>OR</u>

- pushing a swing
- etc.

Generically

$$L[y] = y'' + p(t)y' + q(t)y = g(t).$$

To handle the $g(t)$, we need a "particular solution" $\underline{Y_p(t)}$

s.t. $L[Y_p] = Y_p'' + p\cdot Y_p' + q\cdot Y_p = g(t).$

What about ICs?

How about solution to

$L[y] = 0$? "Homogeneous Eq.".

Let's call ~~hom~~ sol. to homogeneous eq. $Y_H(t)$

$$Y_H(t) = C_1 y_1(t) + C_2 y_2(t).$$

(2 lin. indep sols to $L[y]=0$)

Does $Y_H + Y_p$ satisfy eq. $L[y] = g(t)$?

$$L[Y_H + Y_p] = L[Y_H] + L[Y_p] \quad \text{(L linear)}.$$
$$= 0 + g(t) \checkmark$$

So ... $Y_H + Y_p$ satisfies ODE.

constants used to satisfy ICs

satisfies inhomogeneity.

General solution:

$$y(t) = Y_p(t) + Y_H(t)$$
$$= Y_p(t) + c_1 y_1 + c_2 y_2 .$$

To solve inhomogeneous eq.s:

1. homogeneous solution.
   (what we've been doing so far).
2. particular solution
   to satisfy inhomogeneity.
3. Add together $\rightarrow$ general solution
4. If given ICs, solve for constants.

Start with 2^{nd} order linear odes that are inhomogeneous with constant coefficients.

- Know how to find homog. sol.
- How to find particular sol.?

Method of undetermined coefs

→ to find particular sol.

→ if ODE has const. coefs.

Boils down to making an educated [~~solution~~] guess.

Eg] Find a particular solution to $L[y] = y'' + 3y' + 2y = 3x+1$.

Guess the form of the particular solution

$$y_p(x) = Ax + B$$

where A, B are undetermined.

- Guess form with undetermined coefs
- Plug into ODE to find coefs such that ODE satisfied.

Plugging in: $y_p' = A$, $y_p'' = 0$

$$L[y_p] = (0) + 3(A) + 2(Ax+B) = 3x+1.$$

$$2A \cdot x + (3A+2B) = 3x + 1.$$

Two equations for unknowns A, B:

$$2A = 3 \qquad (x^1)$$

$$3A + 2B = 1 \qquad (x^0).$$

Solving, $A = 3/2$, $B = -7/4$.

So our particular solution is

$$\boxed{y_p(x) = \frac{3}{2}x - \frac{7}{4}.}$$

The homogeneous solution of this eq. is $y_H = C_1 e^{-x} + C_2 e^{-2x}$

General solution is

$$y(x) = \underbrace{C_1 e^{-x} + C_2 e^{-2x}}_{y_H} + \overbrace{\frac{3}{2}x - \frac{7}{4}}^{y_p}.$$

If given ICs, solve for constants here.