Method of undetermined coefficients (continued)

Eg] Find the general solution of

$$L[y] = y'' + 3y' + 2y = e^{3x}.$$

general solution

$$y(x) = y_h(x) + y_p(x)$$

homog. sol. part. sol.

From last time:

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

$$(\,y_h'' + 3y_h' + 2y_h = 0\,)$$

For particular solution, try method of undet. coefs.

Try $y_p(x) = Ae^{3x}$.

Plug in and find A:

$$y_p'' + 3y_p' + 2y_p = e^{3x}$$

becomes $(9Ae^{3x}) + 3(3Ae^{3x}) + 2(Ae^{3x}) = e^{3x}$.

or $\underbrace{\underbrace{20A}\, e^{3x} = \underline{1}}\, e^{3x}$

Therefore, $20A = 1$ or $A = 1/20$.

and $y_p(x) = \frac{1}{20} e^{3x}$.

And the general solution:

$$\boxed{y(x) = \underbrace{C_1 e^{-x} + C_2 e^{-2x}}_{y_h(x)} + \underbrace{\frac{1}{20} e^{3x}}_{y_p(x).}}$$

If we had initial conditions,
NOW is when we'd solve for constants.

<u>Eg</u> Find a particular solution to

$$y'' + 3y' + 2y = \sin x.$$

To obtain this particular sol., use method of undet. coefs.

Try $y_p(x) = A\sin x$.

Plug in: $y_p'' + 3y_p' + 2y_p = \sin x$

$$(-A\sin x) + 3(A\cos x) + 2(A\sin x) = \sin x.$$

$$A\sin x + 3A\cos x = 1\sin x + 0\cdot\cos x$$

Matching coefs:

$$A = 1 \quad \text{and} \quad 3A = 0 \;?!?$$

Try instead: $y_p(x) = A\sin x + B\cos x.$

Plug in: $y_p'' + 3y_p' + 2y_p = \sin x$

becomes

$$(-A\sin x - B\cos x) + 3(A\cos x - B\sin x)$$
$$+ 2(A\sin x + B\cos x) = \sin x.$$

or $(A-3B)\sin x + (B+3A)\cos x = \sin x + 0\cdot\cos x$

Matching coefficients:

$$A - 3B = 1$$
$$3A + B = 0$$

Solving simultaneously, $A = \frac{1}{10}$, $B = \frac{-3}{10}$

and therefore

$$\boxed{y_p(x) = \frac{1}{10}\sin x - \frac{3}{10}\cos x}$$

Eg| For $y'' + 3y' + 2y = \sin x + e^{3x}$, What form of particular solution would you use for the method of undet. coefs.?

Try: $y_p(x) = A\sin x + B\cos x + Ce^{3x}$.

Eg| Find the particular solution of:

$$y'' - 4y = 2e^{2t}.$$

Use method of undet. coefs.

Try $y_p(t) = Ae^{2t}$.

Plug in: $y_p'' - 4y_p = 2e^{2t}$

becomes $(4Ae^{2t}) - 4(Ae^{2t}) = 2e^{2t}$.

$$0 = 2e^{2t} \; ?!?$$

e^{2t} is a HOMOGENEOUS SOLUTION!

$y_h = e^{rt}$, homog. eq. $\boxed{y_h'' - 4y_h = 0}$

becomes $r^2e^{rt} - 4e^{rt} = 0$

$e^{rt} \neq 0, \quad r^2 - 4 = 0$ (char. eq.)

$$r = \pm 2.$$

Therefore $\underline{y_h(t) = c_1 \underbrace{e^{2t}}_{!!!} + c_2 e^{-2t}.}$

How to remedy?

Try: $y_p(t) = Ate^{2t}.$

Plug in: $y_p'' - 4y_p = 2e^{2t}.$

$$(4Ae^{2t} + \cancel{4Ate^{2t}}) - 4(\cancel{Ate^{2t}}) = 2e^{2t}.$$

$$4Ae^{2t} = 2e^{2t}.$$

Matching coefs, $A = \frac{1}{2}.$ $(4A = 2).$

Therefore our part. sol. is

$$\boxed{y_p(t) = \frac{t}{2} e^{2t}}$$

Note: If any term in the trial expression $y_p(t)$ is in the homogeneous solution, try instead $t y_p(t)$.

If THAT's a homogeneous sol.,

try $t^2 y_p(t)$, etc.

To sum up:

Method of undet. coefs for

$$L[y] = ay'' + by' + cy = g(x)$$

$a \neq 0$, b, c constants

(1) Find homog. sol.

(Find y_h s.t. $L[y_h] = 0$)

(2) Determine form of y_p from $g(x)$. (NOT a homog. sol.)

(3) Plug in: $L[y_p] = g(x)$, match coefs.

(4) Solve linear system of equ'ns for coefs.

... thus we obtain the particular solution y_p.

Then the general solution is:

$$y(x) = \underbrace{y_h(x)}_{\text{homog. sol.}} + \underbrace{y_p(x)}_{\text{part. sol.}}$$

$(c_1 y_1(x) + c_2 y_2(x))$.

Variation of Parameters (3.7)

Consider

$$L[y] = y'' + p(t)\, y' + q(t)\, y = g(t).$$

Not necessarily constant coefs!

What we'll do:

- Fundamental solution set $\{y_1(t), y_2(t)\}$ for the corresponding homog. eq. $(L[y_{1,2}] = 0)$.
- Take for our particular solution

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t).$$

∴ want to find u_1 & u_2 such that

$$L[y_p] = g(t).$$

∴ Therefore plug in and <u>force</u>

$$L[y_p] = g(t).$$

<u>Note</u>: 2 unknowns $(u_1(t)$ & $u_2(t))$.

but only 1 constraint $(L[y_p] = g(t))$.

Means we get to impose a second one ☺.