LECTURE 12

Variation of Parameters (cont'd)

Consider

$$L[y] = y'' + p(t)y' + q(t)y = g(t).$$

Assume $y_1(t)$ & $y_2(t)$ satisfy the associated homogeneous equation.

i.e. $L[y_1] = 0, \quad L[y_2] = 0$

& homogeneous solution $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$.

(y_1, y_2 linearly independent $\rightarrow$ form a fundamental solution set)

In the method of variation of parameters, take for the particular solution

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

and find $u_1(t), u_2(t)$ such that

$$L[y_p] = L[u_1y_1 + u_2y_2] = g(t).$$

Let's go!

$L[y_p] = y_p'' + p \cdot y_p' + q \cdot y_p = g.$ (*)

$y_p = u_1 y_1 + u_2 y_2.$ KNOWN.

To plug into (*), need y_p' & y_p''.

(1) y_p'

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$
$$= (u_1 y_1' + u_2 y_2') + (u_1' y_1 + u_2' y_2).$$

Free constraint $\rightarrow$ set $\boxed{u_1' y_1 + u_2' y_2 = 0}$

Why? want 1st order eq. for u_1, u_2.
(also works out nicely).

so $y_p' = u_1 y_1' + u_2 y_2'$.

(2) y_p''.

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''.$$

Now plug into (*). & re-arrange.
(algebra)

$$u_1[\underbrace{y_1'' + p(t)y_1' + q(t)y_1}_{L[y_1]=0}]$$

$$+ u_2[\underbrace{y_2'' + p(t)y_2' + q(t)y_2}_{L[y_2]=0}]$$

$$+ u_1'y_1' + u_2'y_2' = g(t)$$

y_1, y_2 satisfy associated homog. eq.

$$\boxed{u_1'y_1' + u_2'y_2' = g}$$

2 Equations, 2 unknowns:

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g \end{cases}$$

or... ~~fo~~ in matrix form...

$$\underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{M} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ g \end{pmatrix}$$

What's M^{-1}?

$$M^{-1} = \frac{1}{\det(M)}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}$$

$$\det(M) = y_1y_2' - y_1'y_2 = W[y_1, y_2]\ !!$$

Then $\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix}$

and thus.

$$u_1' = \frac{-y_2 \cdot g}{W}, \quad u_2' = \frac{y_1 \cdot g}{W}.$$

Therefore,

$$u_1(t) = -\int^t \frac{y_2(\tilde{t})\, g(\tilde{t})}{W[y_1, y_2](\tilde{t})}\, d\tilde{t}$$

$$u_2(t) = \int^t \frac{y_1(\tilde{t})\, g(\tilde{t})}{W[y_1, y_2](\tilde{t})}\, d\tilde{t}.$$

So we used the homogeneous sols. $\{y_1, y_2\}$ to find the particular sol.

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

where u_1, u_2 as above.

Method of variation of parameters

For $L[y] = y'' + p \cdot y' + q \cdot y = g(t)$.

(1) Find homogeneous solution

i.e. the two linearly independent solutions that satisfy $L[y] = 0$

$(y'' + p \cdot y' + q \cdot y = 0)$.

$\{y_1, y_2\}$

(2) Set the particular sol.

$$y_p = u_1 \cdot y_1 + u_2 \cdot y_2$$

(3) Compute u_1 & u_2 (integrals, matrix form above).

General solution remains:

$$y(t) = \underbrace{c_1 y_1 + c_2 y_2}_{y_h} + y_p.$$

Eg| Find the general solution to $y'' + y = \tan x$ $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$.

(1) Homogeneous solution, satisfying $y_h'' + y_H = 0$.

Let $y = e^{rx}$.

Obtain char. eq. $r^2 + 1 = 0 \Rightarrow r = \pm i$.

And therefore $\sin x, \cos x$ satisfy homog. eq.

$y_1(x) = \sin x, \quad y_2(x) = \cos x$.

(2) Use method of variation of parameters to find part. sol.

Let $y_p = u_1 y_1 + u_2 y_2$.

(3) Computing u_1 & u_2.

u_1 & u_2 must satisfy:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}.$$

So
$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}.$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = - \begin{pmatrix} -\sin x & -\cos x \\ -\cos x & \sin x \end{pmatrix} \begin{pmatrix} 0 \\ \tan x \end{pmatrix}$$

$$(W = -1) \Rightarrow W = y_1 y_2' - y_1' y_2$$
$$= (\sin x)(-\sin x) - (\cos x)(\cos x)$$
$$= -\sin^2 x - \cos^2 x$$
$$= -1.$$

Thus
$$u_1' = \sin x$$
$$u_2' = \frac{-\sin^2 x}{\cos x}.$$

Then $u_1(x) = -\cos x + C_1$ &

(Take $C_1 = 0$ for simplicity).

And $u_2(x) = \int \frac{-\sin^2 x}{\cos x}\, dx$

$= \int (\cos x - \sec x)\, dx$

etc.

$= \sin x - \ln|\sec x + \tan x| + C_2$

(take $C_2 = 0$ for simp.).

$\therefore y_p(x) = u_1 y_1 + u_2 y_2$

$= (-\cos x)\cdot \sin x$

$+ (\sin x - \ln|\sec x + \tan x|)\cdot \cos x$

$= -\cos x \cdot \ln|\sec x + \tan x|.$

And the general solution is:

$y(x) = y_h + y_p.$

or $\boxed{y(x) = c_1 \cos x + c_2 \sin x - \cos x \cdot \ln|\sec x + \tan x|}$