Announcements

- RECALL next homework due <u>next Wed</u>.
- Comment on HW4:

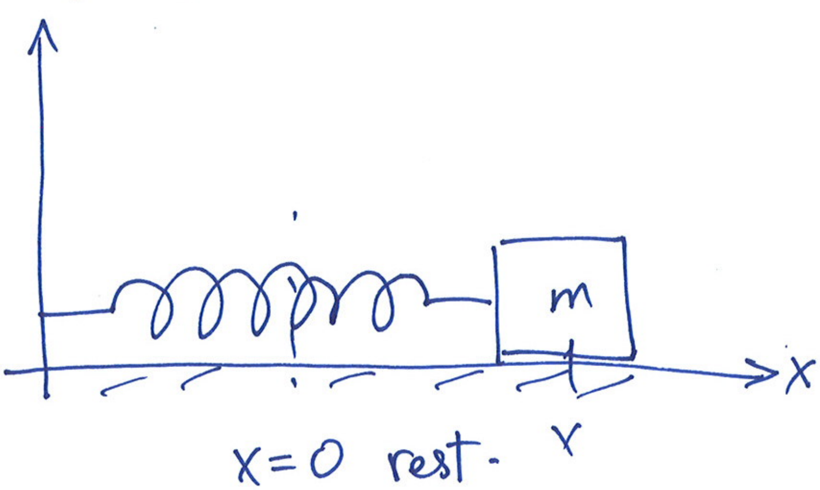
  When using reduction of order to find a second linearly independent solution ...

  → need to <u>show</u> linear independence (or explain it i.e. t and t^2 are linearly independent...)

  (it's <u>why</u> we "ignored constants").

Mechanical Vibrations (3.7):

As discussed



$$\begin{cases} \overset{\text{mass}}{m}x'' + \overset{\text{damping}}{b}x' + \overset{\text{spring const.}}{k}x = 0 \\ x(0) = \text{initial pos.} \\ x'(0) = \text{initial vel.} \end{cases}$$

Recall: Used $x = e^{rt}$ sub to find associated char. eq.

$$mr^2 + br + k = 0.$$

with roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

Different behaviours depending on level of damping.

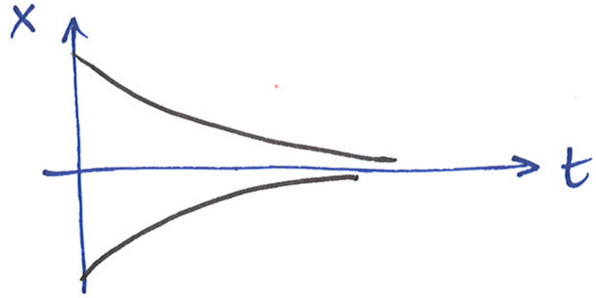
(1) $b > \sqrt{4mk}$

r_1, r_2 real, distinct, negative.

General sol:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Plot:



"Overdamping".

(2) $b = \sqrt{4mk}$

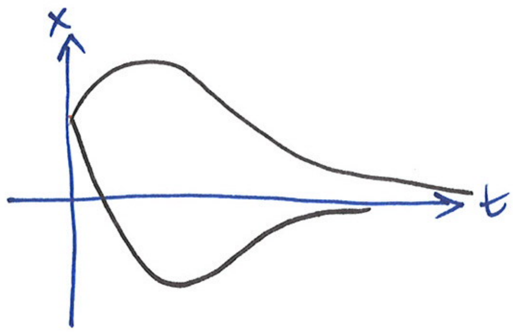
$r_1 = r_2 = r < 0$ (and real)

"repeated root"

General sol:

$x(t) = c_1 e^{rt} + c_2 t e^{rt}$.

Plot:



"Critical damping".

(3) $b < \sqrt{4mk}$

r_1, r_2 complex.

(with -ve real part if $b \neq 0$)

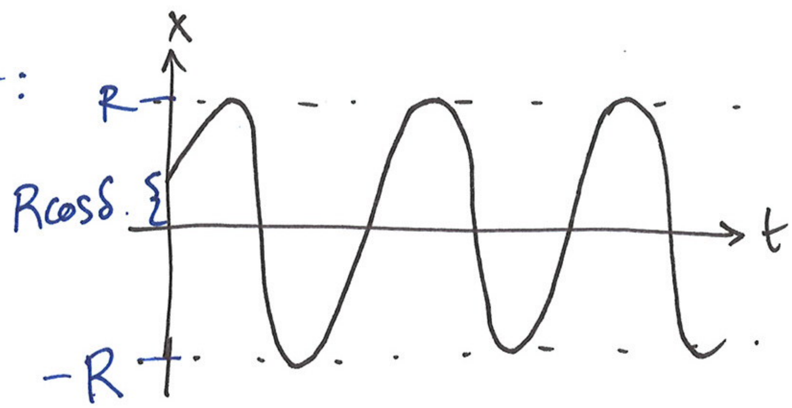
(a) What if $b = 0$?

$$r_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega \quad \left(\omega = \sqrt{\frac{k}{m}}\right).$$

General Sol:

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t).$$

Plot:



Note: can re-write as

$$x(t) = R\cos(\omega t - \delta)$$

amplitude

frequency (natural).

phase shift.

$$x(t) = \underbrace{R\cos\delta}\cos\omega t + \underbrace{R\sin\delta}\sin\omega t$$

vs

$$x(t) = C_1 \cos\omega t + C_2 \sin\omega t.$$

so

$$C_1 = R\cos\delta \qquad R = \sqrt{C_1^2 + C_2^2}$$

and

$$C_2 = R\sin\delta \qquad \delta = \tan^{-1}(C_2/C_1).$$

- $$c_1^2 + c_2^2 = R^2\cos^2\delta + R^2\sin^2\delta$$
$$= R^2(\cos^2\delta + \sin^2\delta)$$
$$= R^2$$

so $R = \sqrt{c_1^2 + c_2^2}$

- $$c_2/c_1 = \frac{\cancel{R}\sin\delta}{\cancel{R}\cos\delta}$$
$$= \cancel{R}\tan\delta$$

so $\delta = \tan^{-1}(c_2/c_1)$.

Definitions:

Frequency ω $\rightarrow$ # oscillations in 2π radians.

Period $T = 2\pi/\omega$ $\rightarrow$ duration of 1 oscillation.

"Undamped oscillations".

(b) $0 < b < \sqrt{4mk}$.

$$r_{1,2} = \frac{-b}{2m} \pm i\underbrace{\frac{\sqrt{|b^2-4mk|}}{2m}}_{\mu} = \frac{-b}{2m} \pm i\mu.$$

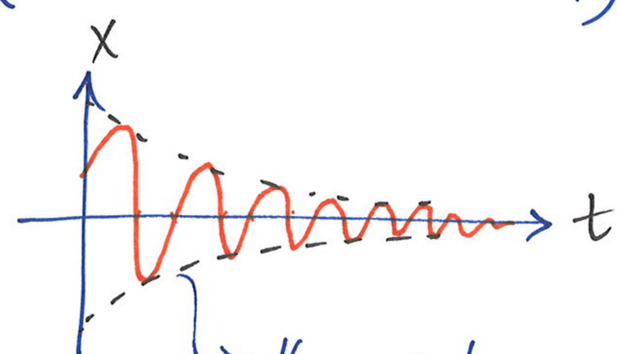
General solution:

$$X(t) = C_1 e^{-\frac{b}{2m}t}\cos(\mu t) + C_2 e^{-\frac{b}{2m}t}\sin(\mu t).$$

OR $\quad X(t) = Re^{-\frac{b}{2m}t}\cos(\mu t - \delta)$

$$\left(\begin{array}{l} R = \sqrt{C_1^2 + C_2^2} \\ \delta = \tan^{-1}(C_2/C_1) \end{array} \right).$$

Plot:



"Damped oscillations".

"envelope of oscillation" $\underline{Re^{-\frac{b}{2m}t}}$.

Definitions:

Quasi-frequency $\mu \rightarrow$
oscillations in 2π radians

Quasi-period $T_d = 2\pi/\mu$.

#2 duration of 1 oscillation.

("Quasi" b/c not periodic motion).

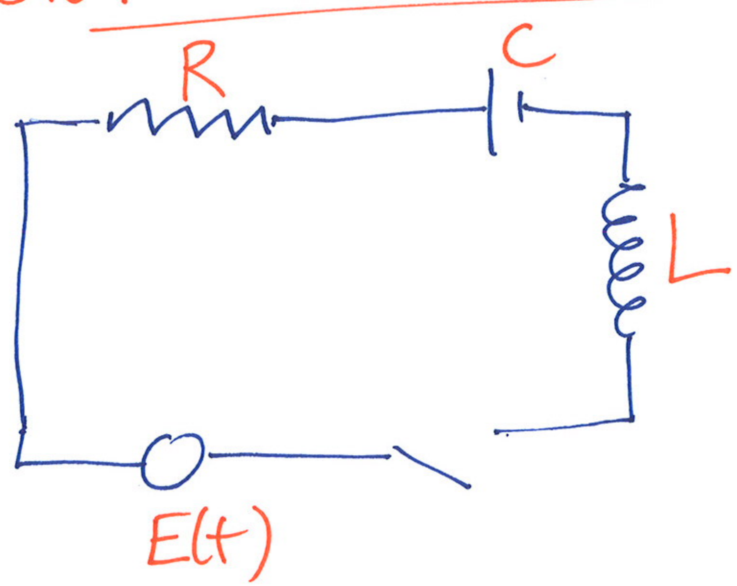
Still call ω (or ω_0) $= \sqrt{k/m}$

"natural frequency".

Forced Vibrations (3.8)

Crystal radio → radio without a battery!

Start with LCR circuit:



I - current (amperes)
R - resistance (ohms)
C - capacitance (Farads)
L - inductance (Henrys)
E - impressed voltage (volts).

Current: $I(t) = \frac{dQ}{dt}$ ← charge.

Kirchoff's 2nd Law:
In a closed circuit, impressed voltage is equal to the sum of voltage drops across the circuit.

Elementary laws of electricity:
→ across resistor IR
→ across capacitor Q/C
→ across inductor $L\,dI/dt$.

From K2,

$$L\frac{dI}{dt} + RI + Q/C = E. \quad (*).$$

Can write as an ODE:

Either

(i) Since $I = dQ/dt$.

$$\begin{cases} LQ'' + RQ' + \frac{1}{C}Q = E \\ Q(0) = Q_0, \quad Q'(0) = I(0) = I_0 \end{cases}$$

(ii) Differentiate (*).

$$\begin{cases} LI'' + RI' + \frac{1}{C}I = E' \\ I(0) = I_0, \quad I'(0) = \dfrac{E(0) - I(0)\cdot R - \frac{1}{C}Q_0}{L} \end{cases}$$

Two second order, linear ODEs for LCR circuit.

(with constant coefs).