Lecture 16

# Announcements

- Midterm next Monday
  - $2^{nd}$ order linear eq.s -
- OH today 4 - 7 SHARP.
- Problem session
  Wed. 6pm Chem. ~~104~~. 124.

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## Beat

Characteristic motion when an oscillator is forced near ~~the~~ its natural frequency.

Illustrate via example:

$$\begin{cases} LQ'' + \frac{1}{C}Q = A\cos\beta t \\ Q(0) = Q'(0) = 0 \end{cases}$$

$\Rightarrow$ all energy from forcing.

Natural frequency

$$\omega^* = \frac{1}{\sqrt{CL}}.$$

Assume $\omega \neq \beta$.

Solving, obtain

$$\begin{aligned} Q &= \left. C_1 \cos(\omega t) + C_2 \sin(\omega t) \right\} \text{homog. sol.} \\ &\quad \left. + \frac{A}{L(\omega^2 - \beta^2)} \cos(\beta t) \right\} \text{part. sol.} \end{aligned}$$

$$Q(0) = Q'(0) = 0$$

so $C_1 = \dfrac{-A}{L(\omega^2 - \beta^2)}$, $C_2 = 0$.

And the solution to our IVP is

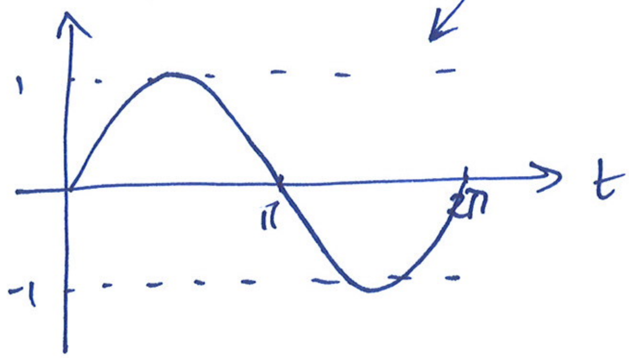
$$Q(t) = \frac{A}{L(\omega^2 - \beta^2)} \left[ \cos(\beta t) - \cancel{\sin} \cos(\omega t) \right].$$

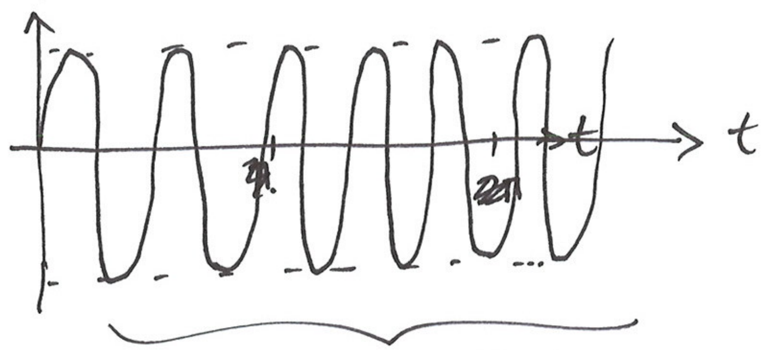
Trig. identity.

$$Q(t) = \frac{2A}{L(\omega^2 - \beta^2)} \sin\left\{ \frac{(\beta + \omega) t}{2} \right\} \sin\left\{ \frac{(\beta - \omega) t}{2} \right\}.$$

Suppose $\omega = 10$, $\beta = 8$.

$$Q = \frac{2A}{36L} \sin t \, \sin 9t .$$

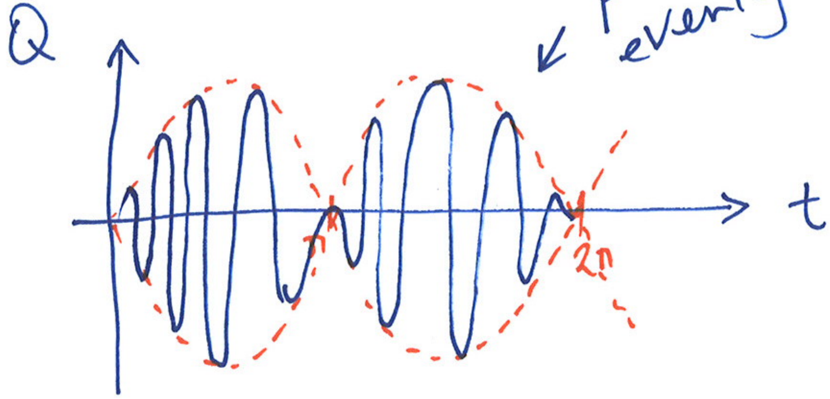



9 oscillations

Combine.

pretend evenly spaced.



# Laplace Transforms

## (Chapter 6).

New approach.

Idea: Use integral transform to turn an IVP into an algebraic expression to be solved. Then solve & transform back.

Especially useful in cases where an inhomogeneity has jump discontinuities.

Eg/
$$\begin{cases} LQ'' + RQ' + \frac{1}{C}Q = \begin{cases} \sin 2t, & 0 \le t < \pi \\ 0, & \pi \le t \le 7\pi/4 \\ \cos 3t, & \frac{7\pi}{4} < t \le 2\pi. \end{cases} \\ Q(0) = Q'(0) = 0 \end{cases}$$

Def'n:

Let $f(t)$ be a function on $[0, \infty]$.

The Laplace transform of $f(t)$ is the function $F(s)$ defined by the integral.

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty f(t) \cdot e^{-st} \, dt.$$

- Domain of $F(s)$: all values for which integral exists.
- Notation: "$F(s)$" AND "$\mathcal{L}\{f\}$".

Note: This is an improper integral

$$\int_0^\infty f(t) e^{-st} \, dt = \lim_{A \to \infty} \int_0^A f(t) e^{-st} \, dt.$$

Eg| Compute the Laplace transform of $f(t) = 1, \quad t \geq 0.$

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{-st} dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}.$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}.$$

Eg| Compute the Laplace transform of $f(t) = e^{at}, \quad t \geq 0.$

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty (e^{at}) e^{-st} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt = \frac{-1}{s-a} e^{-(s-a)t} \Big|_0^\infty$$

$$\boxed{F(s) = \frac{1}{s-a}.}$$

Eg| Try: $f(t) = t, \sin bt, \cos bt.$

(integration by parts).

What about the inverse?

If I know $F(s)$, ~~...~~ how do I get $f(t) = \mathcal{L}^{-1}\{F\}$?

HARD → complex integration.

We'll use tables.

Eg/ What is the inverse transform of $F(s) = \frac{1}{s^{n+1}}$ (n positive integer)

$$\mathcal{L}^{-1}\{F\} = f(t) = \frac{t^n}{n!}$$

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Are there conditions on $f(t)$ for the Laplace transform to exist?

(1) $f(t)$ must be piecewise continuous on $0 \le t \le A$ for any $A > 0$.

(2) $f$ must be of exponential order $a$.

If these are satisfied, $\mathcal{L}\{f\} = F(s)$ exists for some $s > a$.

(Theorem 6.1:2 in text).