LECTURE 17

Theorem:

Suppose that:

(1) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.

(2) $|f(t)| \leq Ke^{at}$ when $t \geq M$. K, a, m constants, $K, M > 0$.

($f(t)$ is of "exponential order")

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

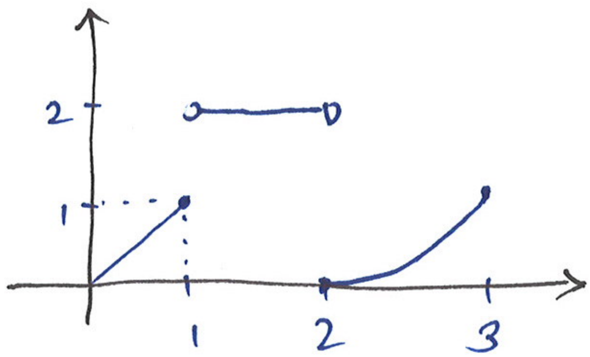
Recall:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Piecewise continuous:

A function $f(t)$ is said to be piecewise continuous on $[a,b]$ if f is continuous at every point in $[a,b]$ except ~~for~~ possibly at a finite # of points at which f has a jump discontinuity.

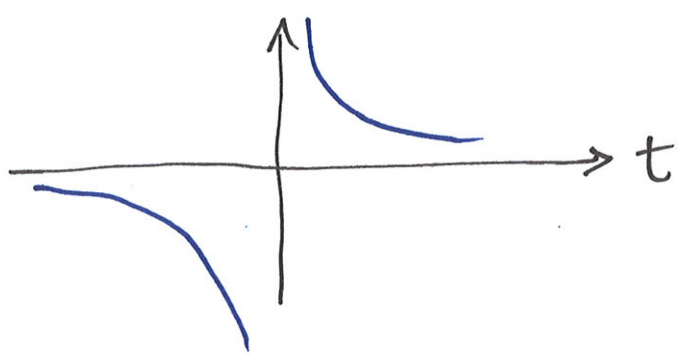
Eg] $f(t) = \begin{cases} t & , 0 \le t \le 1 \\ 2 & , 1 < t < 2 \\ (t-2)^2 & , 2 \le t \le 3 \end{cases}$



is piecewise continuous on $[0,3]$.

note: a function f is "piecewise cts" on $[0,\infty)$ if $f(t)$ is piecewise cts on $[0,N]$ for all $N>0$.

Eg] $f(t) = 1/t$



NOT piecewise continuous on any interval that contains origin!

Exponential order.

A function $f(t)$ is said to be of exponential order if for constants K & $M > 0$

$$|f(t)| \leq Ke^{at} \text{ for } t \geq M.$$

Eg) $e^{5t} \sin 2t$ is of exponential order 5.

$$|e^{5t} \sin 2t| \leq e^{5t} \qquad |\sin\theta| \leq 1.$$

"K" = 1, any M.

Eg) e^{t^2} NOT of exponential order.

$|f(t)| \leq Ke^{at}$ means

$$\frac{|f(t)|}{e^{at}} \leq K.$$

$$\frac{|e^{t^2}|}{e^{at}} = e^{t(t-a)} \longrightarrow \infty \text{ as } t \to \infty.$$

<u>NOT</u> $\leq K$. for any K.

Back to theorem:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

$$= \int_0^M f(t) e^{-st}\, dt + \int_M^\infty f(t) e^{-st}\, dt.$$

$\downarrow$ converges — b/c f is piecewise cts

b/c f of exp. order a.

$$|f(t)e^{-st}| \leq (e^{-st}) \cdot K e^{at}$$

$$\leq K e^{(a-s)t}$$

and $\int_M^\infty e^{(a-s)t}\, dt$ converges

for <u>$s > a$</u>.

Laplace transform is a linear operator.

Let f_1, f_2 be functions whose Laplace transforms exist for $s>a$, and let c be some constant. Then:

- $\mathcal{L}\{f_1+f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}$
- $\mathcal{L}\{cf_1\} = c\mathcal{L}\{f_1\}$

Showing this...

$$\mathcal{L}\{f_1+f_2\} = \int_0^\infty e^{-st}\left(f_1(t)+f_2(t)\right)dt$$
$$= \int_0^\infty e^{-st} f_1(t)\,dt + \int_0^\infty e^{-st} f_2(t)\,dt$$
$$= \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}.$$

$\mathcal{L}\{cf_1\}$ = try yourself.

Solving Initial Value Problems via Laplace transforms.

Eg Solve $\begin{cases} y'' + 4y' - 5y = te^t \\ y(0)=1, \quad y'(0)=0 \end{cases}$

Take Laplace transform of both sides:

$$\mathcal{L}\{y'' + 4y' - 5y\} = \mathcal{L}\{te^t\}$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{te^t\}.$$

Compute transforms:

- $\mathcal{L}\{te^t\} = \int_0^\infty te^t e^{-st}\,dt$

  $= \dfrac{1}{(s-1)^2}$ (integrating by parts).

- $\mathcal{L}\{y\} = \int_0^\infty y(t) e^{-st}\, dt$
$= Y(s).$

- $\mathcal{L}\{y'\} = \int_0^\infty \left(\frac{dy}{dt}\right) e^{-st}\, dt.$

integrate by parts:

$u = e^{-st}$ $\qquad dv = y'\, dt$

$du = -se^{-st}\, dt$ $\qquad v = y.$

so $\mathcal{L}\{y'\} = y \cdot e^{-st} \Big|_0^\infty + s \int_0^\infty e^{-st} y(t)\, dt$

$= -y(0) + sY(s)$

initial data.

☆ Assume $\mathcal{L}\{y\}$ exists.

therefore assuming y of exponential order.

so $\lim_{t \to \infty} y(t) e^{-st} = 0.$

In general,

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots$$

$$\dots - s f^{(n-1)}(0) - f^{(n)}(0).$$

where $\mathcal{L}\{f\} = F(s)$

$f(0), f'(0), \dots, f^{(n)}(0)$ are initial values.

$$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$$

$$= s^2 Y(s) - s$$

as $y(0)=1,\ y'(0)=0.$

Thus the Laplace transform of our ODE is

$$\underbrace{(s^2 Y(s) - s)}_{\mathcal{L}\{y''\}} + 4\overbrace{(sY(s) - 1)}^{\mathcal{L}\{y'\}} - 5Y(s) = \frac{1}{(s-1)^2}.$$

Now what? Solve for $Y(s)$,

take inverse to get $y(t)$.

$$(s^2+4s-5)Y-(s+4)=\frac{1}{(s-1)^2}$$

$$Y(s)=\frac{s^3+2s^2-7s+5}{(s-1)(s+5)(s-1)^2}$$

CHECK.