Announcements

- Midterm MONDAY.
- OH tomorrow (Sat) 2-4pm
  (im Math 104)
- Mid-term surveys today

$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^3(s+5)}$$

(the Laplace transform of the solution of our IVP).

To get $y(t)$ need inverse transform

$$\mathcal{L}^{-1}\{Y(s)\} = y(t).$$

Property of inverse transform:

When inverse exists, it's LINEAR.

That is, for $\mathcal{L}\{f\} = F(s)$, $\mathcal{L}\{g\} = G(s)$, c some constant:

$$\mathcal{L}^{-1}\{F(s) + c\,G(s)\} = \mathcal{L}^{-1}\{F(s)\} + c\,\mathcal{L}^{-1}\{G(s)\}$$
$$= f(t) + c\,g(t)$$

Split up $Y(s)$ using partial fractions decomposition:

$$\frac{s^3 + 2s^2 - 7s + 5}{(s-1)^3(s+5)} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}$$

Obtain: $A = 35/216$, $B = 181/216$,
$C = -1/36$, $D = 1/6$.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^3 + 2s^2 - 7s + 5}{(s-1)^3(s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{35}{216}\cdot\frac{1}{(s+5)} + \frac{181}{216}\cdot\frac{1}{(s-1)} - \frac{1}{36}\cdot\frac{1}{(s-1)^2} + \frac{1}{6}\cdot\frac{1}{(s-1)^3}\right\}$$

$$= \frac{35}{216}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} + \frac{181}{216}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{36}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$
$$+ \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} \quad (\text{linearity of } \mathcal{L}^{-1})$$

~~y(t)=~~ From table:

- $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$
- $\mathcal{L}\{t^n e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$ n a positive integer.

$$\boxed{y(t) = \frac{35}{216}\cdot e^{-5t} + \frac{181}{216}e^{t} - \frac{1}{36}\cdot te^{t} + \frac{1}{12}t^2e^t}$$

Inverting: factoring, partial fractions decomposition, pattern recognition (for tables)... PRACTICE.

Note: on inverting $F(s) = \dfrac{1}{(s-1)^2}$

or computing $\mathcal{L}\{te^t\}$.

options (1) Table (if it's there)

(2) shift or translation property + table.

Translation or shift property

If $\mathcal{L}\{g(t)\} = G(s)$ exists for $s > \alpha$,

$$\mathcal{L}\{e^{at} \cdot g(t)\} = \int_0^\infty e^{at} g(t) e^{-st}\, dt$$

$$= \int_0^\infty g(t)\, e^{-(s-a)t}\, dt$$

$$= \cancel{\mathcal{L}\{G} \; G(s-a).$$

So $\mathcal{L}\{te^t\}$ = "$\mathcal{L}\{t\}$" shifted by 1.

BAD notation do not use.

since $\mathcal{L}\{t\} = 1/s^2$

$$\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$$

Step Functions (6.3):

Mentioned earlier L. Transforms are a nice way to solve IVPs when inhomogeneity has jump discontinuities.

Tool to help with that

⇒ Step function

(aka Heaviside function)

Defined by:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \quad (c > 0).$$

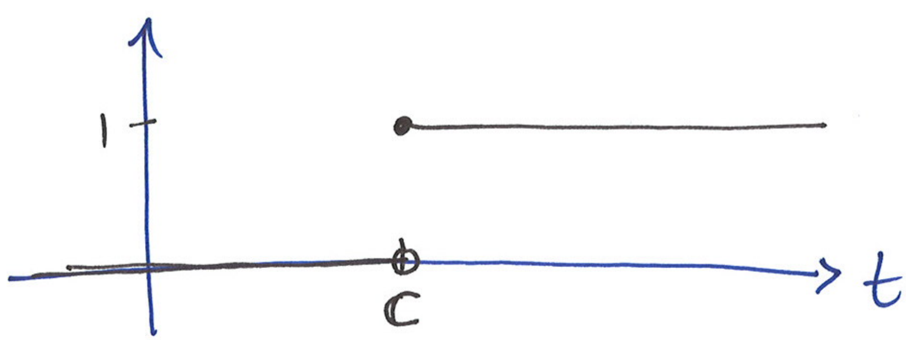
Notation: May also see

- $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

  so $u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} = u_c(t).$

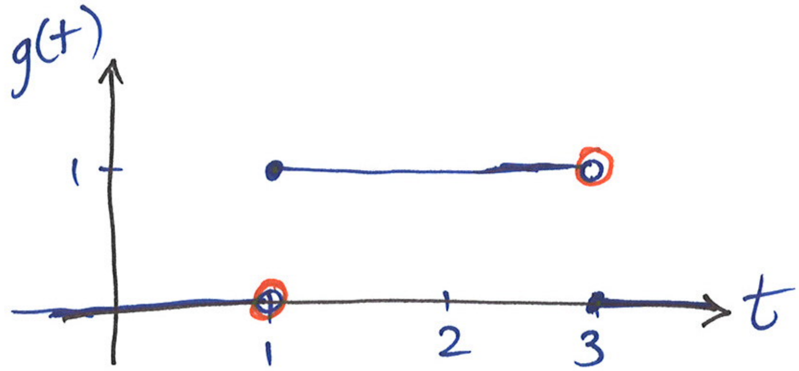
- $H(t)$ (treated same way).

$u_c(t)$:



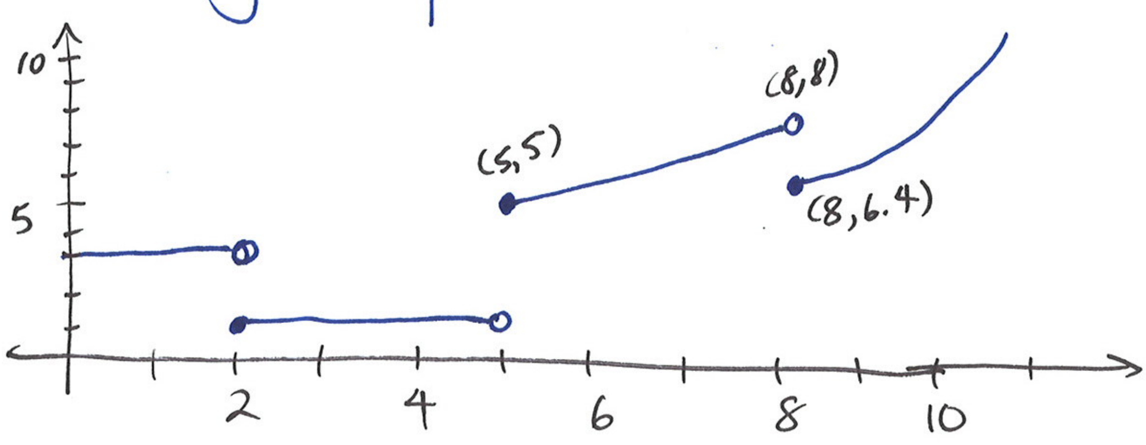
Eg| What does $u_1(t) - u_3(t) = g(t)$ look like? (plot).

$$g(t) = \begin{cases} 0-0=0, & t<1 \\ 1-0=1, & 1 \le t < 3 \\ 1-1=0, & t \ge 3 \end{cases} = \begin{cases} 0, & t<1 \\ 1, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$



Eg| Write $f(t) = \begin{cases} 3, & t<2 \\ 1, & 2 \le t < 5 \\ t, & 5 \le t < 8 \\ t^2/10 & t \ge 8 \end{cases}$

using step functions.



$$f(t) = 3(1-u_2(t)) + 1(u_2(t)-u_5(t)) + t(u_5(t)-u_8(t)) + \frac{t^2}{10}(u_8(t))$$

★USING PREV. EG.

$$f(t) = 3 - 2u_2(t) + (t-1)u_5(t) + \left(\frac{t^2}{10} - t\right)u_8(t)$$

OR

$$f(t) = 3 + \overbrace{(1-3)}^{-2.}u_2(t) + (t-1)u_5(t) + \left(\frac{t^2}{10} - t\right)u_8(t).$$

Laplace transform of Step functions:

(1) $$\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)\,e^{-st}\,dt.$$

$$= \underbrace{\int_0^c u_c(t)e^{-st}\,dt}_{=0} + \int_c^\infty u_c(t)e^{-st}\,dt.$$

RECALL:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

For $c > 0$

$$= \int_c^\infty e^{-st}\,dt = \left.\frac{-e^{-st}}{s}\right|_c^\infty = \frac{e^{-cs}}{s}.$$

$$\boxed{\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}}$$

(2) What about ...

$$g(t) = \begin{cases} 0, & t < c \\ f(t), & t \geq c \end{cases} = f(t)\,u_c(t)$$

What we <u>really</u> want is

$$\mathcal{L}\{f(t)u_c(t)\}.$$

NEXT TIME.