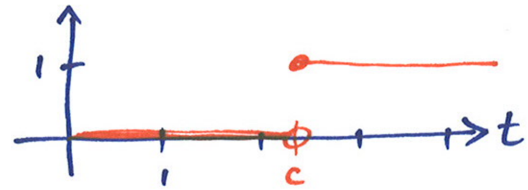
Unit step functions.

$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \geq c \end{cases}$$



Showed $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$.

What about $g(t) = \begin{cases} 0, & 0 \le t < c \\ f(t-c), & t \geq c. \end{cases}$

$$= f(t-c)\,u_c(t).$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{f(t-c)\,u_c(t)\}$$

$$= \int_0^\infty f(t-c)\,u_c(t)\,e^{-st}\,dt.$$

$$= \int_c^\infty f(t-c)\,e^{-st}\,dt \quad \left(\begin{array}{c}\text{as } u_c(t) = 0 \\ \text{for } t < c\end{array}\right)$$

Let $u = t - c$ then $du = dt$
$t = u + c$

$$= \int_0^\infty f(u)\,e^{-s(u+c)}\,du$$

$$= e^{-cs}\int_0^\infty f(u)\,e^{-su}\,du$$

$$= e^{-cs}\,F(s).$$

For $F(s) = \mathcal{L}\{f(t)\}$

$$\boxed{\mathcal{L}\{f(t-c)\cdot u_c(t)\} = e^{-cs}\cdot F(s)}$$

Note: $\mathcal{L}\{f(t)\cdot u_c(t)\} = e^{-cs}\,\mathcal{L}\{f(t+c)\}$.

and $\boxed{\mathcal{L}^{-1}\{e^{-cs}\,F(s)\} = f(t-c)\,u_c(t).}$

Eg1 Compute the Laplace transform of

$$g(t) = \begin{cases} 0, & 0 \leq t < 2 \\ t, & t \geqslant 2. \end{cases} \quad = \quad t\,u_2(t)$$

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t\,u_2(t)\}$$

looks like $\mathcal{L}\{f(t)\,u_2(t)\}$

for $f = t$.

$$\begin{aligned} \mathcal{L}\{g(t)\} &= e^{-2s}\,\mathcal{L}\{t+2\} \\ &= e^{-2s}\left[\mathcal{L}\{t\} + \mathcal{L}\{2\}\right] \quad \mathcal{L} \text{ linear.} \\ &= e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right] \quad \text{(tables)} \end{aligned}$$

$$\mathcal{L}\{g(t)\} = \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}.$$

Eg] Compute the inverse transform of $G(s) = e^{-3s}/s^2+4$.

Let $F(s) = \frac{1}{s^2+4}$. tables.

Then $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\sin(2t) = f(t)$

Since $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u_c(t)$

$$\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s^2+4}\right\} = \frac{\sin[2(t-3)]}{2}u_3(t)$$

Eg] Compute $\mathcal{L}\{\cos(t)\cdot u_\theta(t)\}$ for some $\theta > 0$.

$\mathcal{L}\{\cos t \cdot u_\theta(t)\} = e^{-\theta s}\mathcal{L}\{\cos[t+\theta]\}$

$\cos(t+\theta) = \cos t\cdot\cos\theta - \sin t\sin\theta$.

$\mathcal{L}\{\cos t\cdot u_\theta(t)\} = e^{-\theta s}\mathcal{L}\{\cos t\cos\theta - \sin t\sin\theta\}$

$= e^{-\theta s}\cos\theta\,\mathcal{L}\{\cos t\} - e^{-\theta s}\sin\theta\,\mathcal{L}\{\sin t\}$

($\mathcal{L}$ linear)

$$\mathcal{L}\{\cos t \cdot u_\theta(t)\} = \frac{e^{-\theta s}\cos\theta \cdot s}{s^2+1} - \frac{e^{-\theta s}\cdot \sin\theta}{s^2+1}$$

"

IVPs with Discontinuous Forcing Functions: (6.4)

Start with something simple:

$$\begin{cases} y'' = \begin{cases} t, & 0 \le t < 3 \\ 0, & t \geq 3. \end{cases} \\ y(0) = y'(0) = 0 \end{cases}$$

- Write RHS interms of step fns.
- Transform both sides
- Solve for trans. of solution
- invert to get sol.

  (hard part).

- $\begin{cases} t, & 0 \le t < 3 \\ 0, & t \ge 3. \end{cases} = t - tu_3(t).$

- Transform both sides:

$$\mathcal{L}\{y''\} = \mathcal{L}\{t - tu_3(t)\}$$

$$\Rightarrow \mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$$
$$= s^2 Y(s)$$

$$\Rightarrow \mathcal{L}\{t - tu_3(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{t\, u_3(t)\}$$
$$= \frac{1}{s^2} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}$$

Thus IVP becomes

$$s^2 Y(s) = \frac{1}{s^2} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}.$$

- Solving for $Y(s)$:

$$Y(s) = \frac{1}{s^4} - \frac{e^{-3s}}{s^4} - \frac{3e^{-3s}}{s^3}$$

invert at home.