Lecture 2

HW 1 : due Next Friday
posted on Website
p.15 1(a)&, 4, 8, 13, 18
p.24 18, 20
p.39 5, 14, 24

Office Hours : Wed. 4 – 6pm
+ by appointment.

Population Growth ($N = N(t)$)

$$\begin{cases} \dfrac{dN}{dt} = rN \\ N(0) = N_0 \end{cases}$$

Solution $N(t) = N_0 e^{rt}$.

(to verify, plug into D.E.).

introduce a carrying capacity K.

$$\begin{cases} \dfrac{dN}{dt} = rN(1 - N/K) \\ N(0) = N_0 \end{cases}$$

"Logistic Equation"
(Verhulst 1838)

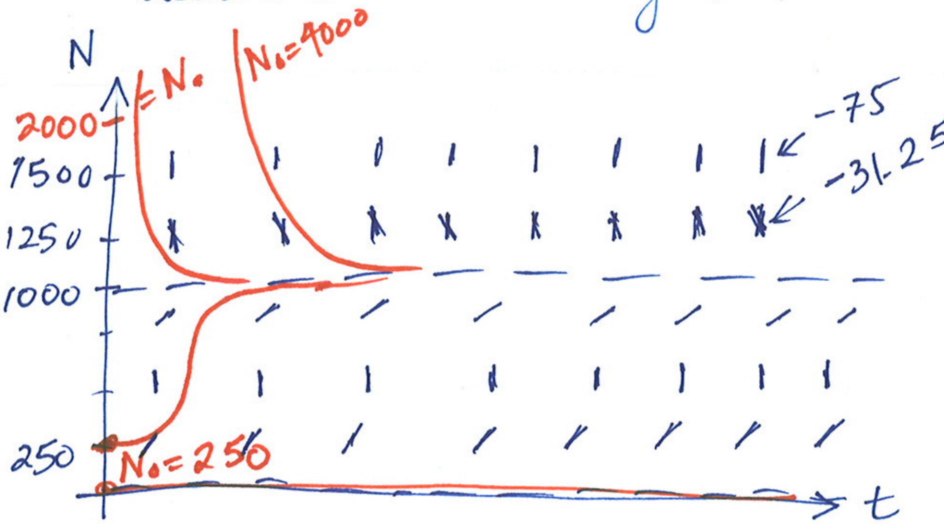
notice now nonlinear.

Definition : An autonomous eq. is one where the indep. var. does not appear explicitly.

Eg) For $y = y(x)$, $\frac{dy}{dx} = f(y)$:

Understand via a <u>direction field</u>.

assume $r = 0.1 \text{ day}^{-1}$, $K = 1000$ individuals.



$N = 250$, $\frac{dN}{dt} = r(250)\left(1 - \frac{250}{1000}\right)$

$= 18.75$.

$N = 500$, $\frac{dN}{dt} = 25$

$N = 750$, $\frac{dN}{dt} = 18.75$

Find equil. points by setting $\frac{dN}{dt} = 0$.

(since <u>autonomous</u>).

$\Rightarrow rN(1 - N/K) = 0$

$N = 0$ or $N = K$.

$= 1000$.

- Can draw direction fields if eq. is not autonomous

eg) $\begin{cases} \frac{dN}{dt} = \frac{r}{(t+1)} N(1 - N/K) \\ N(0) = N_0 \end{cases}$

Can't just set $\frac{dN}{dt} = 0$ to get equil. (flat slope).

Can still understand limiting behaviour as $t \to \infty$

Separable equations
(2.2; 1.2)

Def'n: The 1st order differential equation $\frac{dy}{dx} = f(x,y)$ is SEPARABLE if $f(x,y)$ can be written as $g(x)\,p(y)$

$$\Rightarrow \frac{dy}{dx} = g(x)\,p(y) \text{ is separable.}$$

eg) $\frac{dy}{dx} = \frac{2x + xy}{y^2+1} = x \cdot \left(\frac{2+y}{y^2+1}\right).$

How to solve: $\frac{dy}{dx} = g(x)\,p(y).$

(1) Multiply both sides by dx and $1/p(y)$

$$\Rightarrow \frac{dy}{p(y)} = g(x) \cdot dx.$$

(2) Integrate both sides $\int \frac{dy}{p(y)} = \int g(x)\,dx.$

$\rightsquigarrow$ implicit solution in y
(i.e. <u>not</u> in form $y = f(x)$)

(3) If possible, solve for y.

Eg] $\dfrac{dy}{dx} = \dfrac{x-5}{y^2} \rightarrow \dfrac{dy}{dx} = \underbrace{(x-5)}_{g(x)} \cdot \underbrace{\dfrac{1}{y^2}}_{p(y).}$

- multiply by $dx/p(y)$

$$y^2\,dy = (x-5)\,dx.$$

- integrate both sides

$$\int y^2\,dy = \int (x-5)\,dx.$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

$C/3$

- solve for y: $y = \left(\frac{3}{2}x^2 - \frac{15}{\cancel{3}}x + \tilde{C}\right)^{1/3}$

This is the general solution to the ODE

→ good for <u>any</u> initial condition.

If we had an initial condition $y(a) = y_0$, we could solve for $\tilde{c}$ and obtain a solution to the IVP.

e.g. $y(1) = 0$, say,

$$y(x) = \left(\frac{3}{2}x^2 - 15x + \tilde{C}\right)^{1/3}$$

$$y(1) = 0 = \left(\frac{3}{2} - 15 + \underline{\underline{\tilde{C}}}\right)^{1/3}$$

$$\tilde{C} = 15 - 3/2 = \frac{27}{2}$$

$$\boxed{y(x) = \left(\frac{3}{2}x^2 - 15x + \frac{27}{2}\right)^{1/3}}$$

is the sol. to the IVP $\begin{cases} \dfrac{dy}{dx} = \dfrac{x-5}{y^2} \\ y(1) = 0 \end{cases}$

Eg] mass falling in gravity. Solve the IVP

$$\begin{cases} \dfrac{dv}{dt} = g - \dfrac{\gamma}{m} v \\ v(0) = V_0, \text{ constant} \end{cases}$$

$$\frac{dv}{dt} = \underbrace{\left(g - \frac{\gamma}{m} v\right)}_{p(v)} \cdot \underbrace{1}_{g(t)}$$

Solution : $\boxed{v(t) = \frac{mg}{\gamma} - \left(\frac{mg}{\gamma} - v_0\right) e^{-\frac{\gamma}{m} t}}$