Announcement:

HW7 due Fri. Nov. 5

p. 336 #1, 10

p. 343 # 25

p. 350 # 7, 13, 22, 29.

$$\begin{cases} y'' = \begin{cases} 1, & 0 \le t < 3 \\ 0, & t \geq 3 \end{cases} = 1 - u_3(t) \\ y(0) = y'(0) = 0 \end{cases}$$

- Took Laplace transform of both sides and solved for $\mathcal{L}\{y(t)\} = Y(s)$.

$$Y(s) = \frac{1}{s^4} - \frac{e^{-3s}}{s^4} - \frac{3e^{-3s}}{s^3}.$$

Find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^4} - \frac{e^{-3s}}{s^4} - \frac{3e^{-3s}}{s^3}\right\}.$$

$$= \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}}_{1} - \underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^4}\right\}}_{2} - 3\underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\}}_{3}.$$

$$= 1 - 2 - 3 \cdot 3$$

1 $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$.

Look at table.

use $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

$$= \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$= \frac{1}{3!}t^3 = \boxed{\frac{1}{6}t^3 = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}}$$

2 $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^4}\right\}$

Recall:

- $\mathcal{L}\{f(t-c)u_c(t)\} = e^{-cs} \cdot \mathcal{L}\{f(t)\} = e^{-cs}F(s).$
- $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u_c(t)$ where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$\mathcal{L}^{-1}\left\{e^{-3s}\cdot \underbrace{\frac{1}{s^4}}_{F(s).}\right\}$$

Let $F(s)=\frac{1}{s^4}$; $f(t)=\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$

$$=\frac{1}{6}t^3$$

Therefore,

$$\mathcal{L}^{-1}\left\{e^{-3s}/s^4\right\}=\frac{1}{6}(t-3)^3u_3(t)$$

③ $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\}$.

Let $F(s)=\frac{1}{s^3}$

Then $f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2}t^2$

Therefore $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^3}\right\}=\frac{1}{2}(t-3)^2u_3(t)$

Thus,

$$y(t)=①-②-3\cdot③$$

$$=\frac{1}{6}t^3-\frac{1}{6}(t-3)^3u_3(t)-\frac{3}{2}(t-3)^2u_3(t)$$

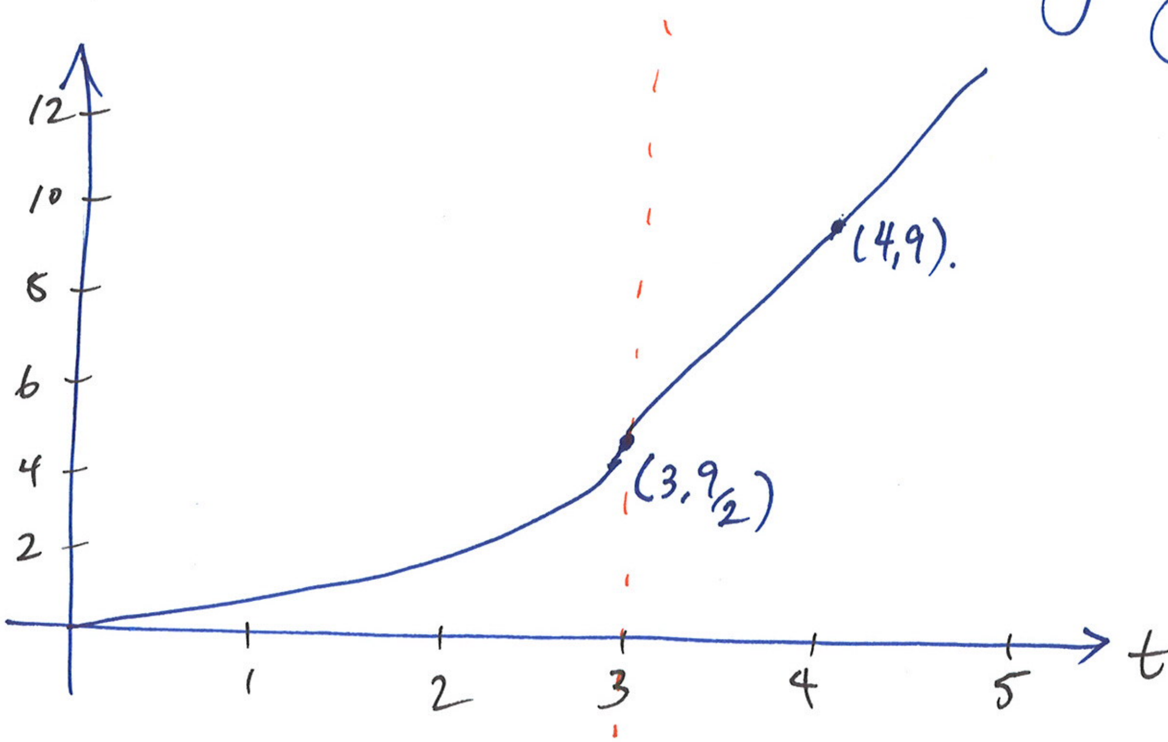
$$\boxed{y(t) = \frac{t^3}{6} - \left[\frac{1}{6}(t-3)^3 + \frac{3}{2}(t-3)^2\right]u_3(t)}$$

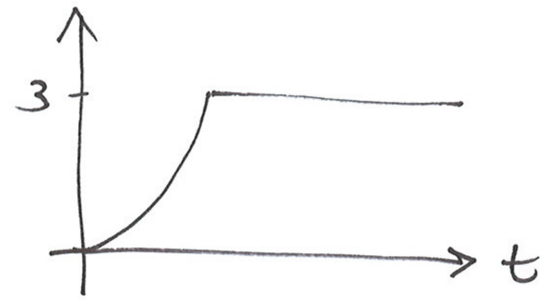
Plot:

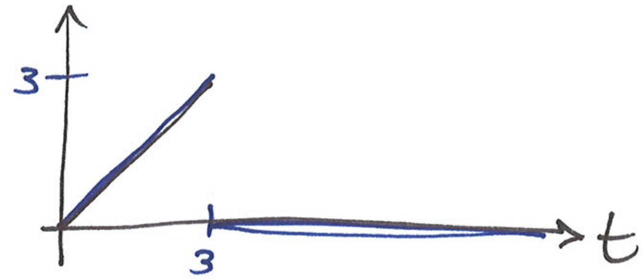
$$y(t) = \begin{cases} t^3/6 & 0 \le t < 3 \\ \frac{t^3}{6} - \left[\frac{1}{6}(t-3)^3 + \frac{3}{2}(t-3)^2\right], & t \geq 3 \end{cases}$$

$$y(t) = \begin{cases} t^3/6 & , \ 0 \le t < 3 \\ \frac{9t}{2} - 9 & , \ t \geq 3 \end{cases}$$

(simplifying)



1st Derivative :  (y')

2nd Derivative  (y'')

Note: Theorem of existence & uniqueness guarantees that the solution $y(t)$ and 1st 2 derivatives are cts, *except* where g is discont. (for $y'' + p(t)y' + q(t)y = g(t)$).

Here, g has a jump discont. at $t=3$.

Notice that at $t=3$, y is cts, y' is cts, but y'' is not!

In general,

for $y'' + p(t)y' + q(t)y = g(t)$ $(*)$

where p, q cts on $\alpha < t < \beta$,

g ~~cts on~~ piecewise cts only,

If $\psi(t)$ is a solution of $(*)$,

ψ, ψ' are continuous, and

ψ'' is discontinuous in same spots as $g(t)$.

Eg) LCR circuit, no resistance.

$$q'' + \beta^2 q = \begin{cases} 0, & 0 \le t < \pi \\ \sin\omega t, & \pi \le t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$$

($\beta = \frac{1}{\sqrt{CL}}$, natural freq.)

($\omega \neq \beta$).

Long problem → handout.

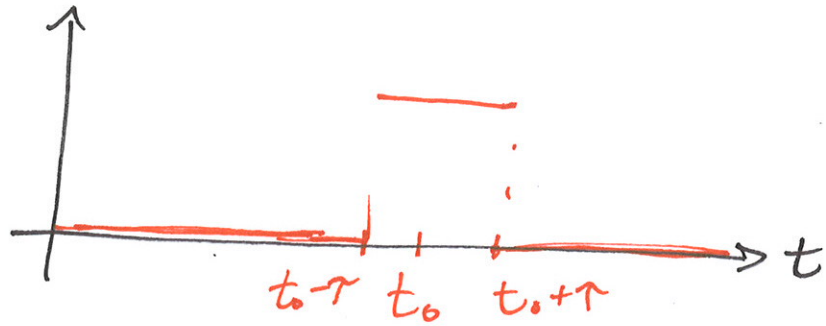
(~~Easy l/w~~ Easier way → next week).

Impulse Functions aka Dirac Delta functions

Say $ay'' + by' + cy = g(t)$

Say $g(t)$ large during some short interval.

$g(t)$:



(τ small)

Measure strength of forcing by integration:

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt = \int_{-\infty}^{\infty} g(t)\,dt.$$

Next time:

- Normalize $I(\tau)$ (so $I(\tau)=1$)
- take limit as $\tau \to 0$

$\Rightarrow \delta(t)$.