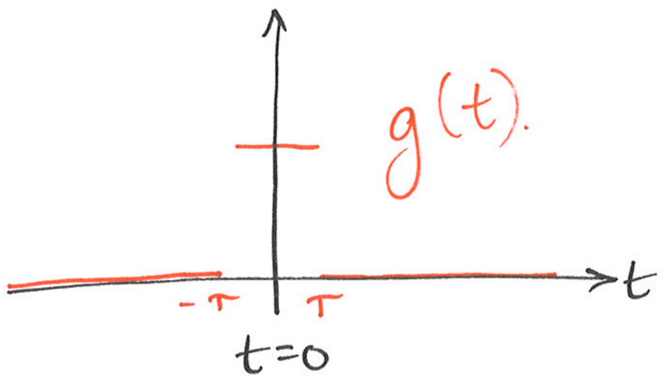
LECTURE 21

Impulses (Dirac Delta Function).

impulse → integral of force
→ change in momentum (roughly speaking).

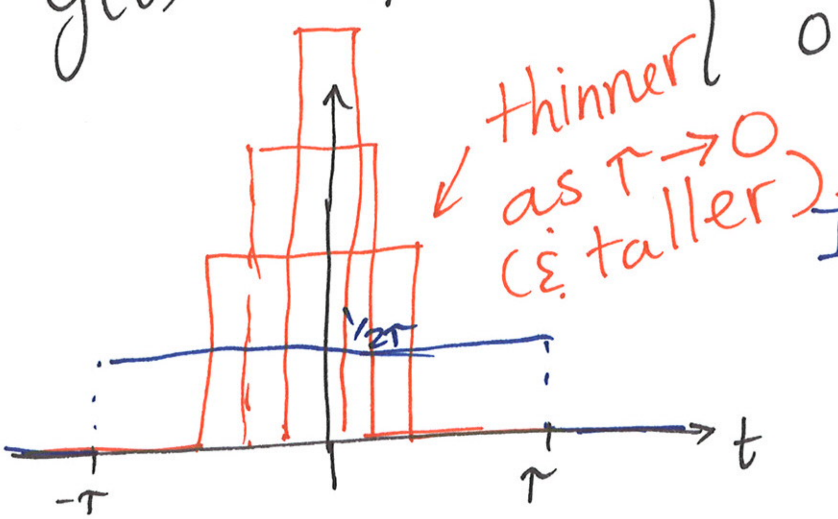


<u>Want</u>

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt.$$

Normalize $g(t)$ s.t. $I(\tau) = 1$.

$$g(t) = d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < 0 < \tau \\ 0, & \text{o/w.} \end{cases}$$



$$I(\tau) = \int_{-\tau}^{\tau} d_\tau(t)\,dt$$
$$= \int_{-\infty}^{\infty} d_\tau(t)\,dt$$
$$= 1.$$

Now take $\tau \to 0$.

$$\lim_{\tau \to 0} d_\tau(t) = 0, \quad \text{for } t \neq 0$$

But since $I(\tau) = 1$ for all $\tau \neq 0$,

follows that $\lim_{\tau \to 0} I(\tau) = 1$.

- Idealized unit impulse function: imparts magnitude 1 impulse at $t = 0$, zero everywhere else.
- Call this limit $\delta(t)$

"Dirac Delta Function"

$$\left(\delta(t) = \lim_{\tau \to 0} d_\tau(t) \right).$$

Properties:

$\delta(t) = 0$ for $t \neq 0$

OR $\delta(t - t_0) = 0$ for $t \neq t_0$

Then $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

or $\displaystyle\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1.$

and $$\int_A^B \delta(t-t_0)\,dt = \begin{cases} 1, & A \le t_0 \le B \\ 0, & \text{o/w.} \end{cases}$$

δ-functions & Laplace Transforms

Assume $t_0 > 0$.

$$\mathcal{L}\{\delta(t-t_0)\} = \cancel{\lim_{\tau\to 0}\int_0^{\infty}} \lim_{\tau\to 0} \mathcal{L}\{d_\tau(t-t_0)\}.$$

$$= \lim_{\tau\to 0} \int_0^{\infty} d_\tau(t-t_0)\, e^{-st}\,dt$$

$$= \lim_{\tau\to 0} \int_{t_0-\tau}^{t_0+\tau} \frac{e^{-st}}{2\tau}\,dt \quad \text{(evaluate).}$$

$$= \lim_{\tau\to 0} \frac{e^{-st_0}}{2\tau s}\left(e^{s\tau} - e^{-s\tau}\right)$$

l'Hôpital's rule.

$$= e^{-st_0}$$

This is a result of the "sifting property" of δ-fns:

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\, dt = f(t_0).$$

(prove using mean value theorem).

Eg] Mass-spring system, impulse at $t=\pi$. (magnitude 3).

$$\begin{cases} \dfrac{d^2y}{dt^2} + 9y = 3\delta(t-\pi) \\ y(0)=1, \; y'(0)=0 \end{cases}$$

(y displacement from rest).

Let $Y(s) = \mathcal{L}\{y(t)\}$.

Take $\mathcal{L}\{\cdot\}$ of both sides:

- $\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0)$.

$$= s^2 Y(s) - s.$$

- $\mathcal{L}\{9y\} = 9Y(s).$
- $\mathcal{L}\{3\delta(t-\pi)\} = 3\mathcal{L}\{\delta(t-\pi)\}$

$$= 3e^{-\pi s}.$$

Thus the transform of our IVP is:

$$s^2 Y(s) - \cancel{s} + 9Y(s) = 3e^{-\pi s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{\cancel{s}s}{s^2+9} + 3\frac{e^{-\pi s}}{s^2+9}.$$

Take inverse to recover $y(t)$:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{3e^{-\pi s}}{s^2+9}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{s^2+9}\right\}.$$

- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos(3t)$

- $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}e^{-\pi s}\right\} =$

Recall : $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u_c(t)$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Let $F(s) = \frac{3}{s^2+9}$.

$f(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin(3t)$

$\left(\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}\right)$

- $\mathcal{L}^{-1}\left\{\frac{3e^{-\pi s}}{s^2+9}\right\} = \sin[3(t-\pi)]\,u_\pi(t)$

$= -\sin(3t)u_\pi(t)$.

Thus,

$$\boxed{y(t) = \cos(3t) - \sin(3t)u_\pi(t)}$$

OR $y(t) = \begin{cases} \cos 3t, & 0 \leq t < \pi \\ \cos 3t - \sin 3t, & t \geq \pi. \end{cases}$

Convolution integral.

Motivational eg

$$\begin{cases} q'' + \beta^2 q = \begin{cases} 0, & 0 \leq t < \pi \\ \sin \omega t, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi. \end{cases} \\ \qquad\quad = \sin \omega t \, [u_\pi(t) - u_{2\pi}(t)] \quad \text{(Fri).} \\ q(0) = q'(0) = 0 \end{cases}$$

Take Laplace trans. of both sides, (letting $Q(s) = \mathcal{L}\{q\}$),

$\mathcal{L}\{q''\} = s^2 Q(s)$ (as $q(0) = q'(0) = 0$)

$\mathcal{L}\{q\} = Q$

$\mathcal{L}\{\sin(\omega t)[u_\pi(t) - u_{2\pi}(t)]\} =$ something big.

Our IVP becomes:

$$s^2 Q(s) + \beta^2 Q(s) = \mathcal{L}\{\sin \omega t\,[u_\pi(t) - u_{2\pi}(t)]\}.$$

Solving for Q:

$$Q(s) = \frac{1}{s^2+\beta^2}\,\mathcal{L}\{\sin(\omega t)\,[u_\pi(t) - u_{2\pi}(t)]\}.$$

From table, we know

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\beta^2}\right\} = \frac{1}{\beta}\sin\beta t.$$

Is there a way around partial fractions?

Yes → convolution integral.