LECTURE 22

Announcements:

Problem session
Tonight 6pm MATX 1100.
(on Laplace Transforms).

The Convolution Integral

Theorem:

If $F(s) = \mathcal{L}\{f(t)\}$ & $G(s) = \mathcal{L}\{g(t)\}$ exist for some $s > a \geq 0$,

then

$$F(s) \cdot G(s) = \mathcal{L}\left\{ \int_0^t f(t-\tau)\, g(\tau)\, d\tau \right\}$$

and $$= \mathcal{L}\left\{ \int_0^t f(\tau)\, g(t-\tau)\, d\tau \right\}.$$

We say

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau \quad \text{and or} \quad \int_0^t f(\tau)g(t-\tau)\,d\tau$$

is the "convolution integral" of f & g.

Notation:

$$h(t) = \underline{f * g} = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

and $\int_0^t f(\tau)g(t-\tau)\,d\tau$.

Showing this: Let

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(u)e^{-su}\,du.$$

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty g(\tau)e^{-s\tau}\,d\tau$$

Then

$$F(s)\cdot G(s) = \left(\int_0^\infty f(u)e^{-su}\,du\right)\left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau\right)$$

$$= \int_0^\infty g(\tau)\left[\int_0^\infty f(u)e^{-s(u+\tau)}\,du\right]d\tau$$

For fixed τ, inner integral is:

$$\int_0^\infty f(u)e^{-s(u+\tau)}\,du$$

Let $t = u + \tau$

so this integral becomes

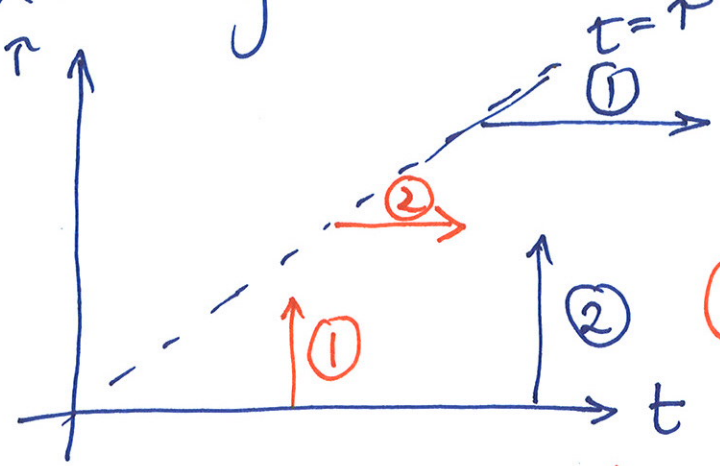
$$\int_\tau^\infty f(t-\tau)e^{-st}\,dt.$$

Putting back in:

$$F(s)\cdot G(s) = \int_0^\infty g(\tau)\left[\int_\tau^\infty f(t-\tau)e^{-st}\,dt\right]d\tau.$$

<u>want</u>: $\int_0^\infty \left(\qquad \right) e^{-st}\,dt.$

Exchange order of integration.



integrate w.r.t. τ first:

1 τ goes from $\underline{0}$ to $\underline{t}$

then 2 t goes from $\underline{0}$ to $\underline{\infty}$

$$F(s)\cdot G(s) = \int_0^\infty \left(\int_0^t f(t-\tau)\, g(\tau)\, d\tau \right) e^{-st}\, dt.$$

(re-arranging).

$$= \mathcal{L}\left\{ \int_0^t f(t-\tau)\, g(\tau)\, d\tau \right\}$$

From def. of Laplace trans. $\blacksquare$

This means if we want

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

where $\mathcal{L}\{f\} = F(s)$, $\mathcal{L}\{g\} = G(s)$.

Eg] Solve the IVP:

$$\begin{cases} y' = t^2 \\ y(0) = 0 \end{cases}$$

using L. transforms.

Still take transform of both sides: Let $Y(s) = \mathcal{L}\{y(t)\}$.

$\mathcal{L}\{y'\} = \mathcal{L}\{t^2\}$ (our eq.)

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$= sY(s).$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

Then
$$sY(s) = \frac{2}{s^3}$$
$$Y(s) = \frac{2}{s^4}.$$

etc.

Instead we'll leave $\mathcal{L}\{t^2\}$ alone.

So $sY(s) = \mathcal{L}\{t^2\}$.

Solving for $Y(s)$:

$$Y(s) = \frac{1}{s} \cdot \mathcal{L}\{t^2\}.$$

Now we want to invert to get $y(t)$. $\mathcal{L}^{-1}\{Y\} = y(t)$.

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} \cdot \mathcal{L}\{t^2\} \right\}$$

Let $F(s) = \frac{1}{s}$, $G(s) = \mathcal{L}\{t^2\}$.

So we have $\mathcal{L}^{-1}\{F(s)\,G(s)\}$

which, using convolution integrals, is

$$\int_0^t f(t-\tau)\, g(\tau)\, d\tau.$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$g(t) = \mathcal{L}^{-1}\{G(s)\}$

Well, $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{1/s\}$

$$= 1.$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

$$= t^2.$$

Then since $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\}$,

$$y(t) = \int_0^t \underbrace{f(t-\tau)}_{=1} \underbrace{g(\tau)}_{\tau^2} d\tau = \int_0^t \tau^2 d\tau$$

$$= \frac{\tau^3}{3}\Big|_0^t$$

$$\boxed{y(t) = t^3/3}$$

$$\left(\underline{\text{OR}} \quad y(t) = \int_0^t \overbrace{f(\tau)}^{=1} \overbrace{g(t-\tau)}^{(t-\tau)^2} d\tau \right.$$

$$\left. = \int_0^t (t-\tau)^2 d\tau \right)$$

$$= t^3/3.$$

Motivational Eg:

We had

$$Q(s) = \underbrace{\mathcal{L}\left\{\frac{1}{\beta}\sin\beta t\right\}}_{F(s)} \cdot \underbrace{\mathcal{L}\left\{\sin(\omega t)\left(u_\pi(t) - u_{2\pi}(t)\right)\right\}}_{G(s).}.$$

Using convolution integrals,

$$q(t) = \mathcal{L}^{-1}\{Q(s)\} = \int_0^t \left(\frac{1}{\beta}\sin[\beta(t-\tau)] \cdot \sin(\omega\tau)[u_\pi(\tau) - u_{2\pi}(\tau)]\right) d\tau.$$

Much nicer than partial fractions expansion!

(Done out → handout online).

Linear Systems (7.5–.9).

Motivating example.

(springs, complicated circuits in text).

Early Epidemic model.

Let $A(t)$ = # asymptomatics at time t.

$Z(t)$ = # ~~infe~~ zombies at time t.

r = rate of infection

n = approx. const. population.

$$\begin{cases} \dfrac{dA}{dt} = rn(A+Z) - \alpha A \\ \dfrac{dZ}{dt} = \alpha A - \delta Z \end{cases}$$

System of 1st order ODEs to understand zombie plague.

What can happen?

$(A, Z) = (0,0)$ is

As $t \to \infty$, A & $Z \to \infty$ – unstable ~~node~~

OR A & $Z \to 0$ – stable.

Note: as A, I get large,

$n \simeq$ const. assumption ~~fails~~ invalid (nonlinear regime)