Announcement

New HW due Fri Nov 12.
(posted later today)

Early-time zombie plague model

$n \approx$ const pop. (uninfected)
r_1 = rate asymptomatics infect
r_2 = rate zombies infect

Had:

$$\begin{cases} \dfrac{dA}{dt} = nr_1 A + nr_2 Z \\ \dfrac{dZ}{dt} = \alpha A - \delta Z. \end{cases}$$

Talked about outcomes.

Re-write as $$\frac{d}{dt}\underbrace{\begin{pmatrix} A \\ Z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} r_1 n & r_2 n \\ \alpha & -\delta \end{pmatrix}}_{\text{matrix } M.} \begin{pmatrix} A \\ Z \end{pmatrix}$$

Looks like $\vec{x}' = M\vec{x}$ ...

type of problem this section focuses on $\longrightarrow$

Linear Systems:

In general, we're interested in systems of the form

$$\vec{x}'(t) = A\,\vec{x}(t). + \vec{F}$$

where $\vec{x}(t)$ is $n \times 1$

A is $n \times n$

$\vec{F}$ is $n \times 1$.

If $\vec{F} = 0 \rightarrow$ homogeneous syst.

$(\vec{x}' - A\vec{x} = 0)$

$\vec{F} \neq 0 \longrightarrow$ non-homogeneous syst. $(\vec{x}' - A\vec{x} = \vec{F})$.

(Focus on homogeneous case for now)

Note: $az'' + bz' + cz = g(t)$ can be written as a 1st order linear system.

Let $z' = y$

Then $y' = z'' = \frac{-b}{a} z' - \frac{c}{a} z + \frac{g}{a}$

$$= \frac{b}{a}y - \frac{c}{a}z + g/a.$$

$$\underbrace{\frac{d}{dt}\begin{pmatrix} z \\ y \end{pmatrix}}_{\vec{x}'(t)} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{b}{a} & -\frac{c}{a} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} z \\ y \end{pmatrix}}_{\vec{x}} + \underbrace{\begin{pmatrix} 0 \\ g/a \end{pmatrix}}_{\vec{F}}.$$

$$\vec{x}'(t) = A\vec{x} + \vec{F}.$$

a lot of the math carries over.

<u>Note</u>: (visualization)

- Can always sketch Solution trajectories vs t.
- 2-dimensional case: <u>phase planes</u> (*) trajectories in x-y space where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Solving homogeneous system:

Want to solve

$$\vec{x}'(t) = A\vec{x}(t).$$

(equilibrium solution (critical point) at $\vec{0} = \vec{x}_e$)

(unless A is "special").

To find homog. Sol →

Pose sub. $\boxed{\vec{x} = \vec{\xi} e^{rt}}$

Then $\vec{x}'(t) = r\vec{\xi} e^{rt}$.

Plug into eq:

$$r\vec{\xi} e^{rt} = A\vec{\xi} e^{rt}$$

<u>or</u> simplify, noting $e^{rt} \neq 0$:

$$\boxed{(A - rI)\vec{\xi} = \vec{0}.}$$

eigenvalues eigenvectors

- eigenvalues are roots of n^{th} order polynomial

$$\det(A - rI) = 0.$$

(A is $n \times n$).

Then solutions are

$$\vec{\xi}_1 e^{r_1 t}, \ \vec{\xi}_2 e^{r_2 t}, \ \text{etc.}$$

Fundamental solution set:

$$\left\{ \vec{X}^{(1)}(t), \vec{X}^{(2)}(t), \cdots, \vec{X}^{(n)}(t) \right\}$$

$$= \{\vec{\xi}_1 e^{r_1 t}, \vec{\xi}_2 e^{r_2 t}, \dots, \vec{\xi}_n e^{r_n t}\}.$$

General solution:

$$\vec{x} = c_1 \cancel{\vec{\xi}_1} \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) + \dots + c_n \vec{x}^{(n)}(t)$$

$$= c_1 \vec{\xi}_1 e^{r_1 t} + c_2 \vec{\xi}_2 e^{r_2 t} + \dots + c_n \cancel{x} \vec{\xi}_n e^{r_n t}.$$

(assuming r_1, r_2 are real & distinct).

Can write general solution

as $$\vec{x} = \underbrace{\Phi(t)}_{\text{fundamental matrix}} \cdot \underbrace{\vec{c}}_{\cancel{\text{mat}}\text{ vector of const.}}$$

(NOT a dot product) ... typo.

$$\begin{pmatrix} \vec{x}^{(1)}(t) & \vec{x}^{(2)}(t) & \cdots & \vec{x}^{(n)}(t) \\ & & & \end{pmatrix}$$

collection of fundamental sols.

$\Phi(t)$ has nice properties (7.7 in text).

Example:

Solve $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}_{A.}\begin{pmatrix} x \\ y \end{pmatrix}$.

Find eigenvalues & eigenvectors of A.

Eigenvalues:

$\det(A - rI) = 0$

$$\det\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix} = 0$$

$$(r+2)^2 - 1 = 0$$

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0.$$

Therefore $r_2 = -3$ & $r_1 = -1$ are e-vals.

<u>Eigenvectors:</u>

For r_1, $\vec{\xi}_1$ s.t. $(A - r_1 I)\vec{\xi}_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{\xi}_1 = 0$$

$$\text{so } \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $r_2 = -3$, $\vec{\xi}_2$ s.t. $(A - r_2 I)\vec{\xi}_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{\xi}_2 = 0$$

$$\text{so } \vec{\xi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The fundamental solution set is therefore:

$$\{\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)\} = \left\{\begin{pmatrix}1\\1\end{pmatrix}e^{-t}, \begin{pmatrix}1\\-1\end{pmatrix}e^{-3t}\right\}.$$

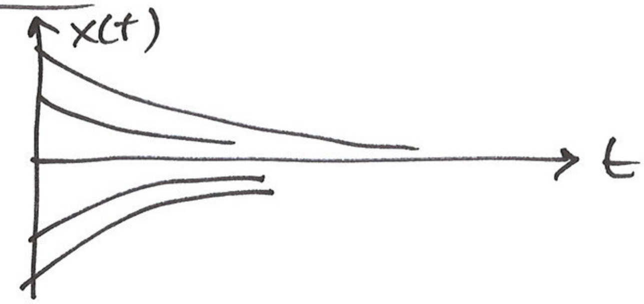
All solutions can be constructed from fundamental sol. set.

$$\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$$

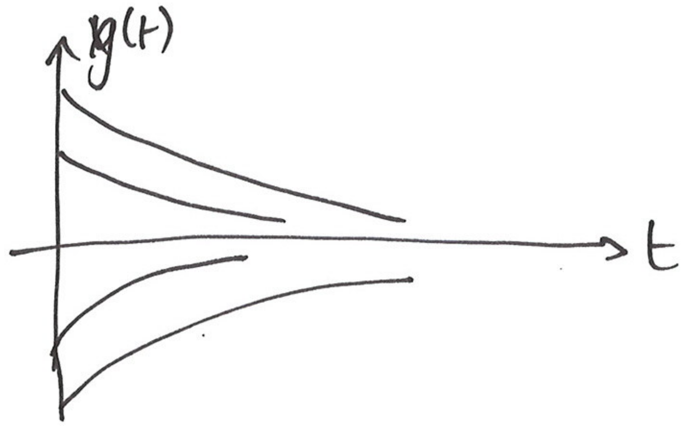
$$= c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-3t} = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}.$$

Solution trajectories:

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}$$

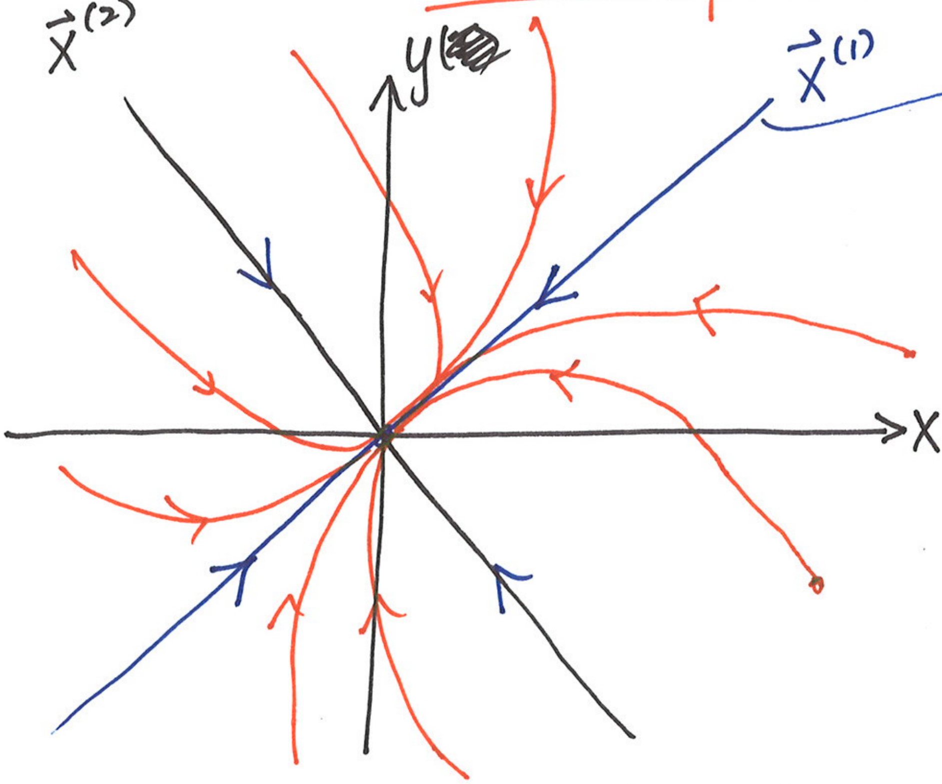


$$y(t) = c_1 e^{-t} - c_2 e^{-3t}$$



x & y are coupled! interesting to visualize how they relate to each other

→ **Phase portrait** ☆



- slope is $\frac{\Delta y}{\Delta x} = 1$
- y-intercept is 0.

① Draw in fundamental sols.

② Fill in other solutions.

As $t \to \infty$, $e^{-3t} \to 0$ <u>faster</u> than $e^{-t} \to 0$