Linear Systems:

$$\vec{x}'(t) = A\vec{x}(t).$$

$n\times 1$ vector

$n\times n$ matrix.

Solve via $\vec{x}(t) = \vec{\xi} e^{rt}$ sub, find $\vec{\xi}$ & r satisfy $A\vec{\xi} = r\vec{\xi}$

r: e-val; $\vec{\xi}$: e-vect. of A.

(obtain $r, \vec{\xi}$ by satisfying

$$(A - rI)\vec{\xi} = \vec{0}$$

for I the identity matrix).

In the 2×2 case we'll characterize critical (equil.) points (s.t. ~~xx~~ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ s.t.

$$\left.\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}\right|_{\substack{x=a \\ y=b}} = 0 \text{ for any } t\,)$$

according to the eigenvalues of A, which reveals the solution behaviour around the crit. point (made clear in phase plane).

What can you get?

- real, distinct e-vals
- complex e-vals
- repeated e-vals

... sound familiar?

Note: E-vals are roots of eq. $\det(A - rI) = 0 \rightsquigarrow$ "characteristic polynomial" or "characteristic eq.".

Friday example:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}_{A.} \begin{pmatrix} x \\ y \end{pmatrix}$$

A has eigenvalues

$$r_1 = -1, \qquad r_2 = -3$$

with associated eigenvectors

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

showed fund. sols, etc on Fri.

general sol:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2 real, distinct, -ve ~~solut~~ eigenvals:

As $t \to \infty$, $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (critical point).

"asymptotically stable".

stable: all trajectories tend to critical point as $t \to \infty$.

unstable: most trajectories go away from critical point as $t \to \infty$.

~~This~~ (0, 0) is also a ~~n~~ "improper node" → most trajectories tangent to $y = x$ line near critical point.

$\therefore$ ~~"CASE 4~~ The critical point (0,0) is an asymptotically stable, improper node.

Always the case for real, negative, distinct e-vals in 2×2 case.

"

- If eigenvalues real, distinct, POSITIVE

$\longrightarrow$ asymptotically ~~stable~~ UNSTABLE improper node.

EG. Solve linear syst.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- draw phase portrait showing trajectories of all solutions.
- Find sol. if IC $x(0)=1, y(0)=3$

System can be written as

$$\vec{x}'(t) = A\vec{x}(t) \quad \text{where } \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}.$$

Pose sub $\vec{x} = \vec{\xi} e^{rt}$

and obtain $(A - rI)\vec{\xi} = \vec{0}$.

<skip steps — lin.alg.>.

A has eigenvalues

$$r_1 = 3, \qquad r_2 = -2$$

with associated eigenvectors

$$\vec{\xi}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Fundamental ~~system~~ solution set:

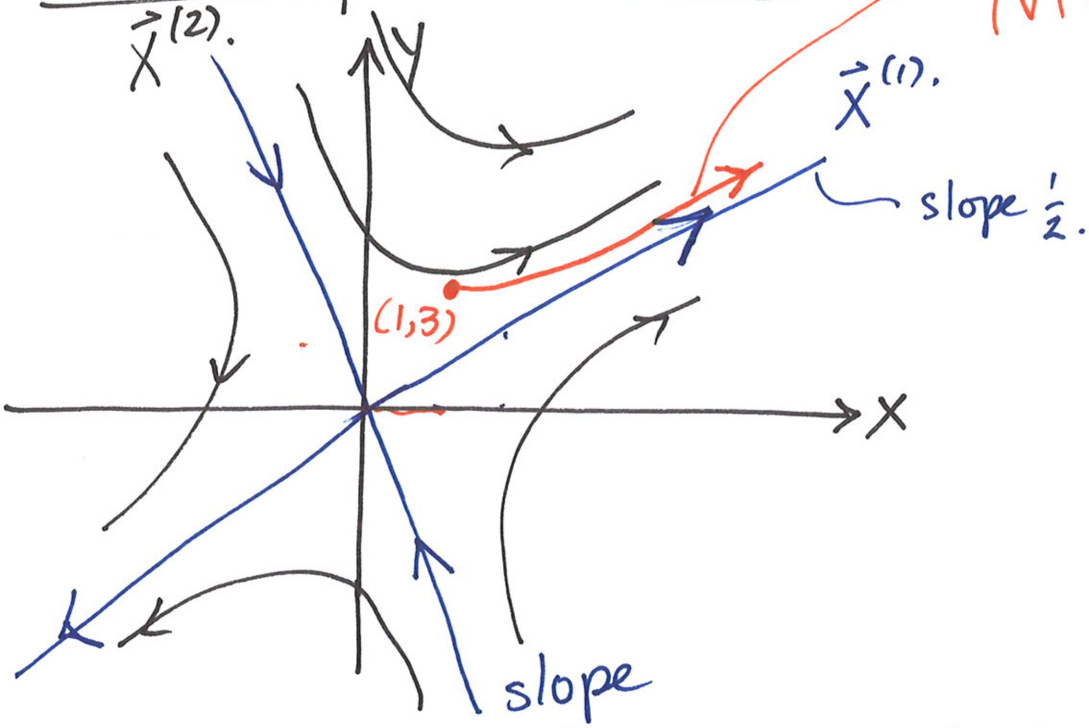
$$\left\{ \vec{x}^{(1)}(t), \vec{x}^{(2)}(t) \right\} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} \right\}$$

General sol. is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)}$$

$$= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}.$$

Phase portrait:



IVP:

IC at

$x(0) = 1$

$y(0) = 3$

① Draw in fundamentals sols.

$\vec{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$, $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$.

clearly linearly indep.

② Fill in approximate traj.

Critical point $(0,0)$ is a "(unstable) saddle point" (or node).

In fact:

2 real, distinct e-vals of opposite sign, critical point is a saddle node, always.

Find sol. if $x(0) = 1$, $y(0) = 3$.

$$x(t) = 2C_1 e^{3t} + C_2 e^{-2t}$$

$$x(0) = \boxed{2C_1 + C_2 = 1}$$

$$\text{\&} \quad y(t) = C_1 e^{3t} - 2C_2 e^{-2t}$$

$$y(0) = \boxed{C_1 - 2C_2 = 3}$$

Solving, $C_1 = 1$, $C_2 = -1$

$$\text{\&} \quad \begin{cases} x(t) = 2e^{3t} - e^{-2t} \\ y(t) = e^{3t} + 2e^{-2t} \end{cases}$$

Eg] Same problem as before

for $\begin{cases} x' = x + y \\ y' = -x + y \\ x(0) = 4, \ y(0) = 0 \end{cases}$

Can write this as

$\vec{x}'(t) = A\vec{x}(t)$ where

$$\vec{x}(t) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Now, the matrix A is associated with

e-vals : $r_{1,2} = 1 \pm i$

and corresponding e-vectors

$$\vec{\xi}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$(r_1 = 1 + i) \qquad (r_2 = 1 - i)$

Could ~~write~~ sol. as

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(1+i)t} + c_2 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1-i)t}.$$

But not very practical →
want real-valued solution.