We had:

$$\begin{cases} \vec{x}'(t) = A\vec{x}, \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \\ \vec{x}(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \end{cases}$$

Solve via $\vec{x}(t) = \vec{\xi} e^{rt}$ sub

turns out, e-vects of A

turns out, e-vals of A.

A has eigenvalues

$$r_1 = 1 + i, \quad r_2 = 1 - i$$

with associated eigenvectors

$$\vec{\xi}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Want real-valued solution.

Let $\vec{\xi}_1 = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{a}} - i \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{b}}$, $\vec{\xi}_2 = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{b}}$.

Then can write 1st fundamental solution

$$\vec{x}^{(1)} = \vec{\xi}_1 e^{r_1 t} = (\vec{a} - i\vec{b}) e^{\overbrace{(1+i)}^{r_1 = 1+i.}t}$$

$$= e^t (\vec{a} - i\vec{b})[\cos t + i \sin t]$$

(recall $e^{i\theta} = \cos\theta + i\sin\theta$).

$$= e^t[\vec{a}\cos t + \vec{b}\sin t]$$
$$+ ie^t[\vec{a}\sin t - \vec{b}\cos t]$$

Similarly,

$$\vec{x}^{(2)} = e^t[\vec{a}\cos t + \vec{b}\sin t]$$
$$- ie^t[\vec{a}\sin t - \vec{b}\cos t]$$

(Complex conjugates!)

Still complex - prefer real-valued solutions!

Instead for our fundamental solutions use:

$$\vec{u}(t) = e^t[\vec{a}\cos t + \vec{b}\sin t] \qquad \left(\begin{array}{c}\text{looks like}\\ \vec{x}^{(1)} + \vec{x}^{(2)}\end{array}\right)$$

$$= \begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix} \quad (\text{see } \vec{a} \;\&\; \vec{b} \text{ above})$$

$$\vec{v}(t) = e^t[\vec{a}\sin t - \vec{b}\cos t] \qquad \left(\begin{array}{c}\text{looks like}\\ \vec{x}^{(1)} - \vec{x}^{(2)}\end{array}\right)$$

$$= \begin{pmatrix} -e^t \cos t \\ e^t \sin t \end{pmatrix}$$

(so $\vec{x}^{(1)} = \vec{u} + i\vec{v}$, $\vec{x}^{(2)} = \vec{u} - i\vec{v}$).

(Can show $\vec{u}, \vec{v}$ are linearly indep.)

New fundamental sol. set is

$$\{\vec{u}, \vec{v}\} = \left\{ e^t[\vec{a}\cos t + \vec{b}\sin t],\ e^t[\vec{a}\sin t - \vec{b}\cos t] \right\}$$

~~(covers any so~~

(linear combination of these gives you any sol, just like $\vec{x}^{(1)}$ & $\vec{x}^{(2)}$)

The general solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 e^t \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$

Note: in general, if e-vals are $r = \lambda \pm i\mu$ (λ, μ real) with associated eigenvectors $\vec{a} \pm i\vec{b}$.

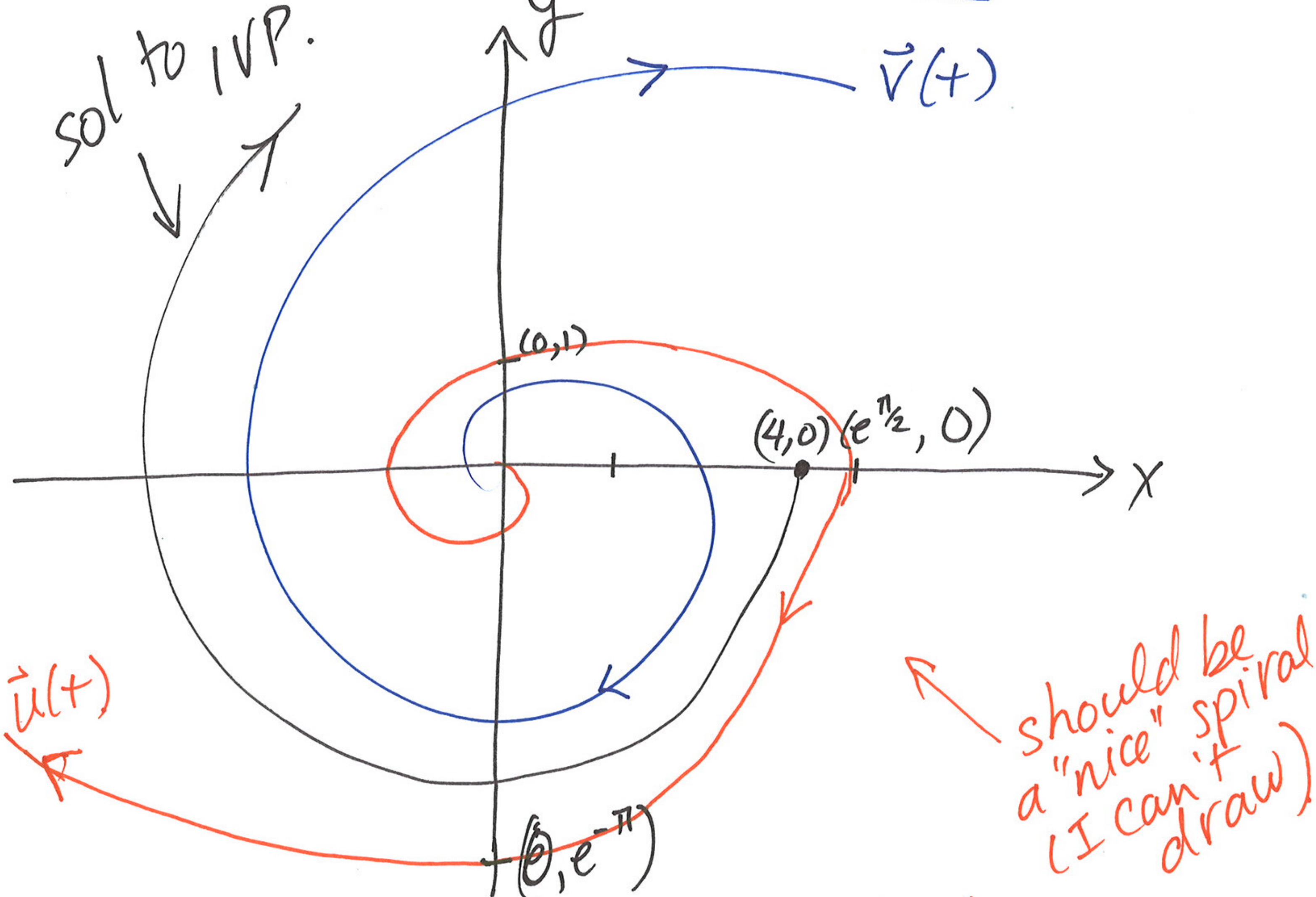
The fundamental sol. set can be written as:

$$\{\vec{u}, \vec{v}\} = \left\{ e^{\lambda t}[\vec{a}\cos\mu t + \vec{b}\sin\mu t], \; e^{\lambda t}[\vec{a}\sin\mu t - \vec{b}\cos\mu t] \right\}.$$

The general sol is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{\lambda t}[\vec{a}\cos\mu t + \vec{b}\sin\mu t] + C_2 e^{\lambda t}[\vec{a}\sin\mu t - \vec{b}\cos\mu t].$$

Draw Phase Plane:



① Draw in fundamental sols.

$$\vec{u}(t) = e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

$$\vec{u}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \vec{u}\left(\tfrac{\pi}{2}\right) = \begin{pmatrix} e^{\pi/2} \\ 0 \end{pmatrix}, \vec{u}(\pi) = \begin{pmatrix} 0 \\ -e^{\pi} \end{pmatrix}$$

② Fill in approximate traj.

Note:

Given: $r_{1,2} = \lambda \pm i\mu$ with

$\lambda > 0 \rightarrow$ critical point is an "unstable spiral"

$\lambda < 0 \rightarrow$ critical pt is a "stable spiral".

Finally: Solve IVP

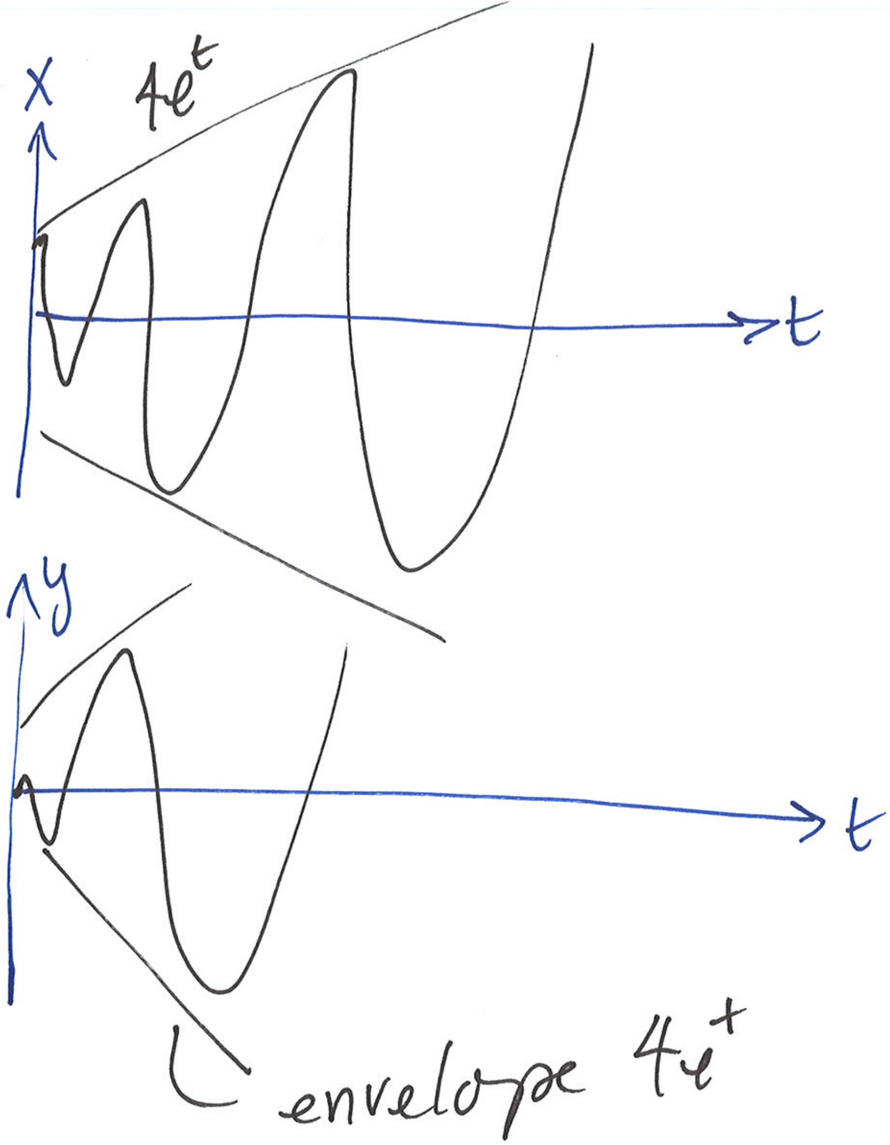
$x(0) = 4, \; y(0) = 0.$

Obtain $C_1 = 0, \; C_2 = -4$

Thus solution of IVP is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4e^t \cos t \\ -4e^t \sin t \end{pmatrix}$$

or $x(t) = 4e^t \cos t$, $y(t) = -4e^t \sin t$.



Repeated e-vals

When the matrix A from the system

$$\vec{x}'(t) = A\vec{x}$$

has repeated e-vals.

Possibilities:

① eigenvalues are associated with distinct, linearly independent eigenvectors. (eg. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$).

② Only one ~~l.~~ eigenvector.

Eg) $\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ x(0) = -2, \; y(0) = 5 \\ \alpha \text{ some real } \# \end{cases}$

Same as before.

Let $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$.

A has repeated e-val $r = \alpha$.

Eigenvectors $\vec{\xi}$ solve

$$(A - \alpha I)\vec{\xi} = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{\xi} = 0.$$

Then $\vec{\xi}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

<u>OR</u> $\vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

are lin. indep. evects.

We'll use the second ones.

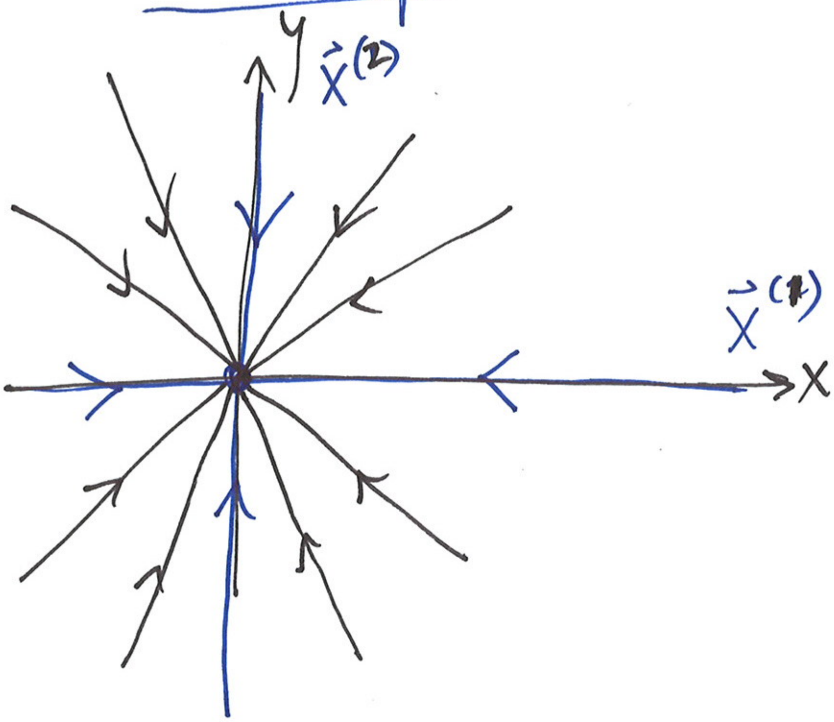
The fundamental sol. set is

$$\left\{\vec{x}^{(1)}, \vec{x}^{(2)}\right\} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\alpha t}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\alpha t}\right\}.$$

General Sol is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\alpha t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\alpha t}.$$

Phase portrait:



① Draw in fund. sols. (Assume $\alpha < 0$)

② Fill in.

Critical point (0,0) is a stable, proper node.

"stable star"

all trajectories are distinct near crit. point.

(if $\alpha > 0$, unstable).