LECTURE 26

# Aside: purely imaginary e-vals

Eg $\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}}$

The matrix A has e-vals $\pm i = r_{1,2}$ with associated e-vects $\vec{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The fundamental solutions are:

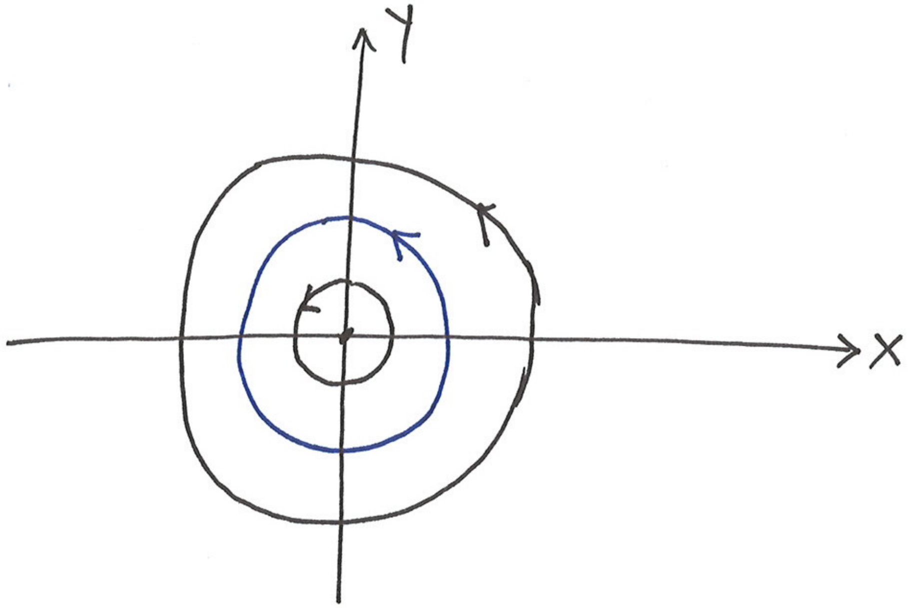
$$\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t = \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

General solution is $\vec{x} = c_1 \vec{u} + c_2 \vec{v}$

or $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$.

# Phase portrait:




① Draw in fund. sols.

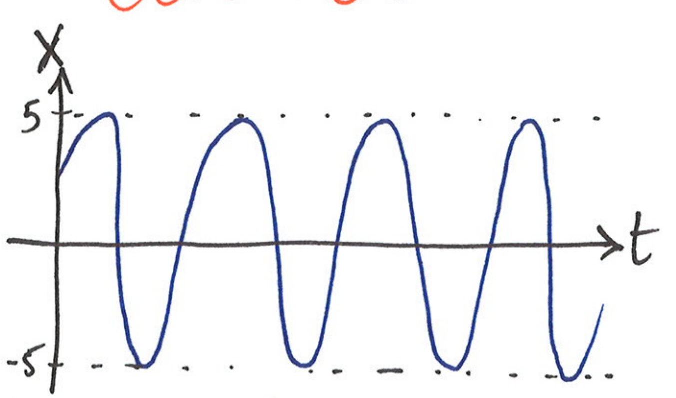
- lie on same line!
- get directions from $\vec{u}, \vec{v}$.

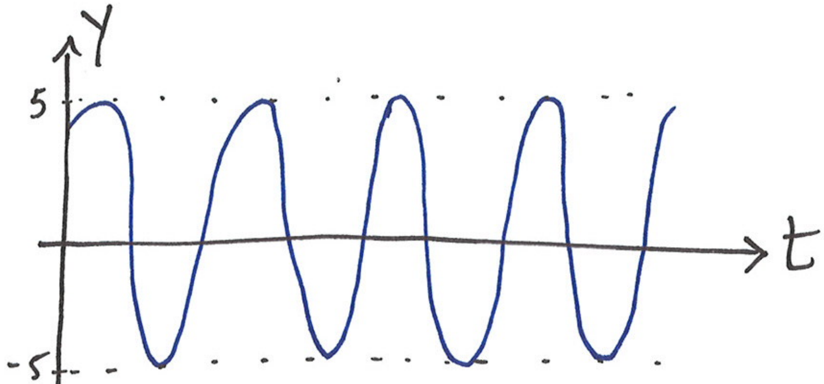
$\vec{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\vec{u}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ etc.

All trajectories form ellipses (in this case, more specifically, circles).

Critical point (0,0) is a "stable centre".




(For $x(0) = 3$, $y(0) = 4$)

REGULAR OSCILLATIONS.

Back to what we were looking at → repeated eigenvalues.

Eg Solve the linear system:

$$x'(t) = -2x - y$$
$$y'(t) = x - 4y$$

Can write interms of matrices/vectors.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} \quad (\vec{x}' = A\vec{x}).$$

The matrix A has char. eq:

$$r^2 + 6r + 9 = 0$$
$$(r+3)^2 = 0$$

roots are e-vals

means $r = -3$ is our repeated root.

Corresponding eigenvectors $\vec{\xi}$ satisfy $(A - rI)\vec{\xi} = \vec{0}$.

here, $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \vec{\xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Only <u>one</u> linearly indep eigenvector $\vec{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

so, <u>one</u> of the fundamental sols is $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$.

What's the other?

When looking at $2^{nd}$ order linear systems, used reduction of order to show the second solution is $te^{rt}$.

Tempted to try $\vec{x} = t\vec{\xi}e^{-3t}$. doesn't quite work.

Instead assume

$$\boxed{\vec{x}^{(2)}(t) = \vec{\xi}te^{-3t} + \vec{\eta}e^{-3t}.}$$

Plug in:

$$\frac{d}{dt}\left(\vec{\xi}te^{-3t} + \vec{\eta}e^{-3t}\right) = A\left(\vec{\xi}te^{-3t} + \vec{\eta}e^{-3t}\right).$$

$$\underbrace{(A\vec{\eta} + 3\vec{\eta} - \vec{\xi})}_{=0}e^{-3t} + \underbrace{(A\vec{\xi} + 3\vec{\xi})}_{=0}te^{-3t} = 0$$

(2) (1)

(1) $(A + 3I)\vec{\xi} = 0. \rightarrow \vec{\xi} = \begin{pmatrix}1\\1\end{pmatrix}$ is the e-vect from before.

② $(A+3I)\vec{\eta} = \vec{\xi}$.

$\vec{\eta}$ here is called the generalized e-vect.

Find $\vec{\eta}$: $\vec{\eta}$ satisfies.

$$(A+3I)\vec{\eta} = \vec{\xi}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

scalar eq. $\eta_1 - \eta_2 = 1$.

Let $\eta_2 = k$, arbitrary.

$\eta_1 = 1 + k$

so $\vec{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Second solution.

$$\vec{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

$k\begin{pmatrix}1\\1\end{pmatrix}e^{-3t}$ linearly dependent with $\vec{x}^{(1)}$... so can ignore $(k=0)$.

$$\vec{x}^{(2)} = \begin{pmatrix}1\\1\end{pmatrix}te^{-3t} + \begin{pmatrix}1\\0\end{pmatrix}e^{-3t}.$$

Fundamental solution set is

$$\left\{ \underbrace{\begin{pmatrix}1\\1\end{pmatrix}e^{-3t}}_{\vec{x}^{(1)}},\ \underbrace{\begin{pmatrix}1\\1\end{pmatrix}te^{-3t} + \begin{pmatrix}1\\0\end{pmatrix}e^{-3t}}_{\vec{x}^{(2)}} \right\}$$

$\therefore$ General solution

$$\begin{pmatrix}x\\y\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-3t} + c_2\left[\begin{pmatrix}1\\1\end{pmatrix}te^{-3t} + \begin{pmatrix}1\\0\end{pmatrix}e^{-3t}\right]$$

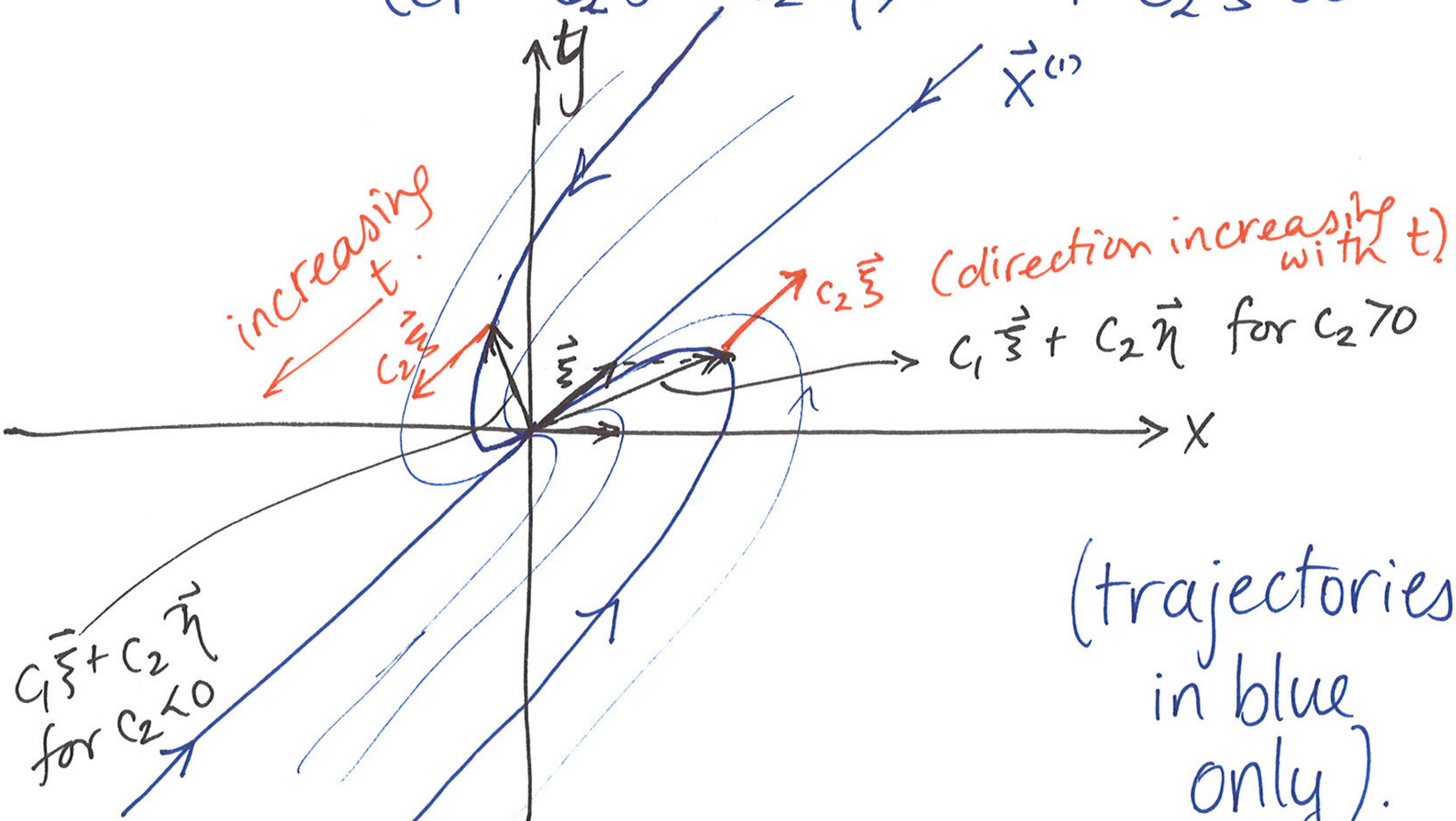
<u>OR</u>

$$\begin{cases} x(t) = c_1e^{-3t} + c_2(te^{-3t} + e^{-3t}) \\ y(t) = c_1e^{-3t} + c_2te^{-3t}. \end{cases}$$

# Draw phase portrait:

$$\vec{x}(t) = c_1\vec{\xi}e^{-3t} + c_2(\vec{\xi}t + \vec{\eta})e^{-3t}$$

$$= (c_1 \cancel{+ c_2}\vec{\xi} + c_2\vec{\eta})e^{-3t} + c_2\vec{\xi}te^{-3t}.$$




(trajectories in blue only).

① Draw in $\vec{x}^{(1)}$.

② $c_1\vec{\xi} + c_2\vec{\eta}$ for $c_2 > 0$ (identify direction as $t$ increases.

As $t \to \pm\infty$, $\approx$ parallel w/ $\vec{\xi}$.

③ Repeat for $c_2 < 0$

④ Sketch in approximate trajectories.

"Stable, improper node"

(if repeated e-val were positive, ~~UNstable proper~~ node.) UNSTABLE, IMPROPER.

# Non-homogeneous Systems:

$$\vec{x}' = A\vec{x} + \vec{F}(t).$$

- if $\vec{F}(t)$ is not a constant vector, system not autonomous, ~~a~~ so no critical points or phase plane diagrams.

[(if $\vec{F}(t)$ is a constant, only difference with work so far is critical pt ~~not~~ moved $(\neq (0,0))$)

ASIDE.]

- Same sol. techniques as with $2^{nd}$ order linear ODEs:
  - → find homog. sol'n
    $$\vec{x}_h'(t) = A\,\vec{x}_h(t)$$
    (eigenvalues/eigenvects)
  - → find particular sol $\vec{x}_p$.
  - → general sol. is
    $$\vec{x} = \vec{x}_h + \vec{x}_p.$$

To find part. sol:

- undet. vectors
- variation of vectors

also: diagonalization, Laplace transforms
(won't cover → in text.)

Ex| Find the general sol. of

$$\cancel{x'(t)=}\ \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 9t \\ 2e^{t} \end{pmatrix}.$$

Know homogeneous sol
(see prev. prob).

$$\vec{X}_h = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]e^{-3t}.$$

Particular sol → use undetermined vectors.

$$\vec{x}_p' = A\vec{x} + \begin{pmatrix} 9t \\ 2e^t \end{pmatrix}$$

[cf. $y'' + p(x)\cdot y' + q(x)\cdot y = k_1 t + k_2 e^t$.]

[Trial: $y_p = At + B + Ce^t$.]

First re-write inhomogeneous part – pull out t-dep.

$$\vec{x}_p' = A\vec{x} + \underbrace{\begin{pmatrix} 9 \\ 0 \end{pmatrix}}\,t + \underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}\,e^t$$

Trial guess:

$$\vec{x}_p(t) = \vec{a}t + \vec{b} + \vec{c}e^t.$$

plug in & find unknown $\vec{a}$, $\vec{b}$, $\vec{c}$. (posted online).