Announcements

- Problem session tonight
  MATX 1100
  6pm

- Midterm <u>Friday</u>

Undet Vectors

Eg | Find the general solution of

$$\begin{cases} \dfrac{dx}{dt} = 3y + 6e^{-t} \\ \dfrac{dy}{dt} = -x - 4y \end{cases}$$

Write as $\frac{d}{dt}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 0 & 3 \\ -1 & -4 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}\underline{e^{-t}}$

- Find homogeneous solution $\vec{x}_h(t)$ s.t. $\vec{x}_h'(t) - A\vec{x}_h(t) = 0$.

Need eigenvals & eigenvects of A.

A has e-vals $r_1 = -1$ & $r_2 = -3$

with corres. e-vects $\vec{\xi}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ & $\vec{\xi}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

So homog. sol. is

$$\vec{x}_h(t) = c_1 \vec{\xi}_1 e^{r_1 t} + c_2 \vec{\xi}_2 e^{r_2 t}$$
$$= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}.$$

Now find particular sol.
Use method of undet. coef (vectors).

$\vec{x}_p$ that satisfies

$$\vec{x}_p'(t) = A\vec{x}_p(t) + \underbrace{\begin{pmatrix} b \\ 0 \end{pmatrix}}_{\vec{g}} e^{-t}$$

Note that $r_1 = -1$ is an e-val-
so inhomogeneity has same
t-dep. (e^{-t}) as homog. sol.

So choose trial expression:

$$\vec{x}_p(t) = \vec{a}\cdot te^{-t} + \vec{b}e^{-t}.$$

Plug in: $\vec{x}_p'(t) = A\vec{x}_p(t) + \vec{g}e^{-t}$

$$\vec{a}e^{-t} - \vec{a}te^{-t} + \vec{b}e^{-t}$$

$$= A\vec{a}te^{-t} + A\vec{b}e^{-t} + \vec{g}e^{-t}.$$

$$\overbrace{(A\vec{a} + \vec{a})}^{=0}te^{-t} +$$

$$\underbrace{[A\vec{b} - \vec{a} + \vec{b} + \vec{g}]}_{=0}e^{-t} = 0.$$

Then

- $A\vec{a} + \vec{a} = 0$ or $\boxed{(A+I)\vec{a} = 0}$ ①

- $A\vec{b} - \vec{a} + \vec{b} + \vec{g} = 0$

  or $\boxed{(A+I)\vec{b} = \vec{a} - \vec{g}}$ ②

Finding $\vec{a}$ & $\vec{b}$ ...

① $(A+I)\vec{a} = 0$

$\vec{a}$ is the eigenvector.

$$\vec{a} = \begin{pmatrix} -3\alpha \\ \alpha \end{pmatrix}, \quad \alpha \text{ to be determined.}$$

② $(A+I)\vec{b} = \vec{a} - \vec{g}$

$$\underbrace{\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}}_{A+I} \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{\vec{b}} = \underbrace{\begin{pmatrix} -3\alpha \\ \alpha \end{pmatrix}}_{\vec{a}} - \underbrace{\begin{pmatrix} 6 \\ 0 \end{pmatrix}}_{\vec{g}}$$

Obtain 2 scalar equations...

- $b_1 + 3b_2 = -3\alpha - 6$.
- $-b_1 - 3b_2 = \alpha$ or $b_1 + 3b_2 = -\alpha$.

For this to have a solution,

NEED $-3\alpha - 6 = -\alpha$.

$$\boxed{\alpha = -3}$$

~~Then~~

Faster way: use left eigen vector.

$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ such that

$$\vec{w}^T A = \lambda A,$$

left e-vect

e-val.

(careful with complex case).

Then $\vec{w}^T(A + rI) = \vec{0}$ ~~in this case~~.

Well, we have

$$(A+I)\cdot\vec{b} = \vec{a} - \vec{g}$$

$$\underbrace{\vec{w}^T(A+I)}_{=0}\vec{b} = \vec{w}^T(\vec{a}-\vec{g})$$

$$\vec{w}^T(\vec{a}-\vec{g}) = 0$$ ← get α from this. "solvability condition".

<u>Here</u> $\overbrace{(w_1 \quad w_2)}^{\vec{w}^T}\overbrace{\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}}^{A+I.} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Obtain $\vec{w}^T = (1 \quad 1)$ or $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

So. $\vec{a} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}$ (since $\alpha = -3$).

We had $b_1 + 3b_2 = 3$

Let $b_2 = k$ arbitrary; $b_1 = 3 - 3k$.

and so $\vec{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + k\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$

Particular solution:

$$\vec{x}_p = \underbrace{\begin{pmatrix} 9 \\ -3 \end{pmatrix}}_{\vec{a}} te^{-t} + \underbrace{\left[\begin{pmatrix} 3 \\ 0 \end{pmatrix} + k\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right]}_{\vec{b}.} e^{-t}.$$

Recall $\vec{x}_H = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$

same. for simplicity, let $k=0$.

Then $\vec{x}_p = \begin{pmatrix} 9 \\ -3 \end{pmatrix} te^{-t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} e^{-t}.$

And general solution is

$$\vec{x} = \vec{x}_H + \vec{x}_p$$

$$\boxed{\vec{x} = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 9 \\ -3 \end{pmatrix} te^{-t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} e^{-t}}$$

Variation of Vectors:

$$\vec{x}' = A(t)\vec{x} + \vec{g}(t)$$

works for non-constant A.

Find fundamental matrix for homogeneous system.

$$\vec{x}_h' = A\vec{x}_h$$

(recall: fundamental matrix = matrix where each col. is fundamental sol.)

Fundamental matrix $\Psi(t)$.

Then as with variation of params, let $\vec{x}(t) = \Psi(t) \cdot \vec{u}(t)$

homog. sol. unknown part.

and find $\vec{u}(t)$ s.t. our eq.

$$\vec{x}' = A(t)\,\vec{x} + \vec{g}(t)$$

is satisfied.

Plug in:

$$\frac{d}{dt}\left[\Psi(t)\,\vec{u}(t)\right] = A(t)\,\Psi(t)\,\vec{u} + \vec{g}(t).$$

$$\Psi'\vec{u} + \Psi\vec{u}\,' = A\Psi\vec{u} + \vec{g}.$$

$$\Psi'(t) = A\Psi.$$

Since $\vec{x}_h\,' = A\vec{x}_h$, each col. of Ψ is an $\vec{x}_h$.

$$\cancel{A\Psi\vec{u}} + \Psi\vec{u}\,' = \cancel{A\Psi\vec{u}} + \vec{g}$$

$$\boxed{\Psi\vec{u}\,' = \vec{g}.}$$

Integrating,

$$\vec{u}'(t) = \Psi^{-1}(t)\vec{g}(t)$$

$$\vec{u}(t) = \int \Psi^{-1}(t)\,\vec{g}(t)\,dt + \vec{c}.$$

Then since $\vec{x} = \Psi\vec{u}$

$$\boxed{\vec{x}(t) = \Psi\vec{c} + \Psi(t)\int_{t_0}^{t}\Psi^{-1}(s)\,\vec{g}(s)\,ds.}$$

Finally, given $\vec{x}(t_0) = \vec{x}_0$.

$$\vec{x}(t_0) = \vec{x}_0 = \Psi\vec{c}\ \ldots$$

$$\vec{c} = \Psi^{-1}(t_0)\,\vec{x}_0.$$

$$\boxed{\begin{aligned}\vec{x}(t) &= \Psi(t)\Psi^{-1}(t_0)\vec{x}_0\\ &\quad + \Psi(t)\int_{t_0}^{t}\Psi^{-1}(s)\,\vec{g}(s)\,ds.\end{aligned}}$$