Nonlinear Systems

(Chap. 9)

Systems of the form

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \quad \text{(2-d system)}$$

Where f & g contain terms like x^2, xy, y^3, etc. (nonlinear terms).

CAN'T write as matrix.

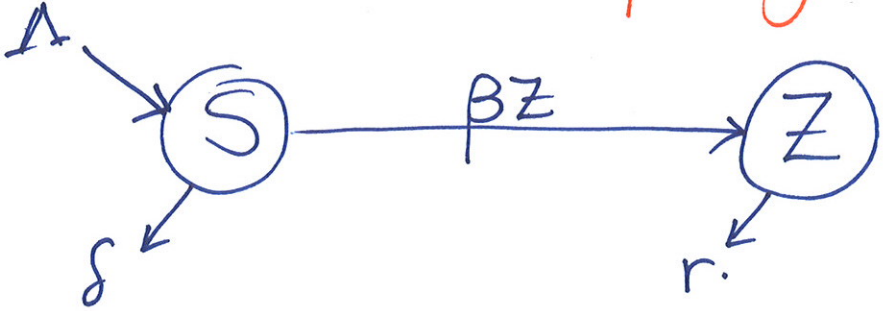
Sometimes write as

$$\vec{x}' = \vec{f}(\vec{x})$$

Note: This is an AUTONOMOUS system ⇒ f, g don't depend on t.

Motivating Example

→ Zombie plague.



S = susceptibles

Z = zombies.

(Based on epidemic models

→ Kermack & Mckendrick, 1920s).

$$
\begin{cases}
\dfrac{dS}{dt} = \Lambda - \beta S Z - \delta S \\[2ex]
\dfrac{dZ}{dt} = \beta S Z - r Z
\end{cases}
$$

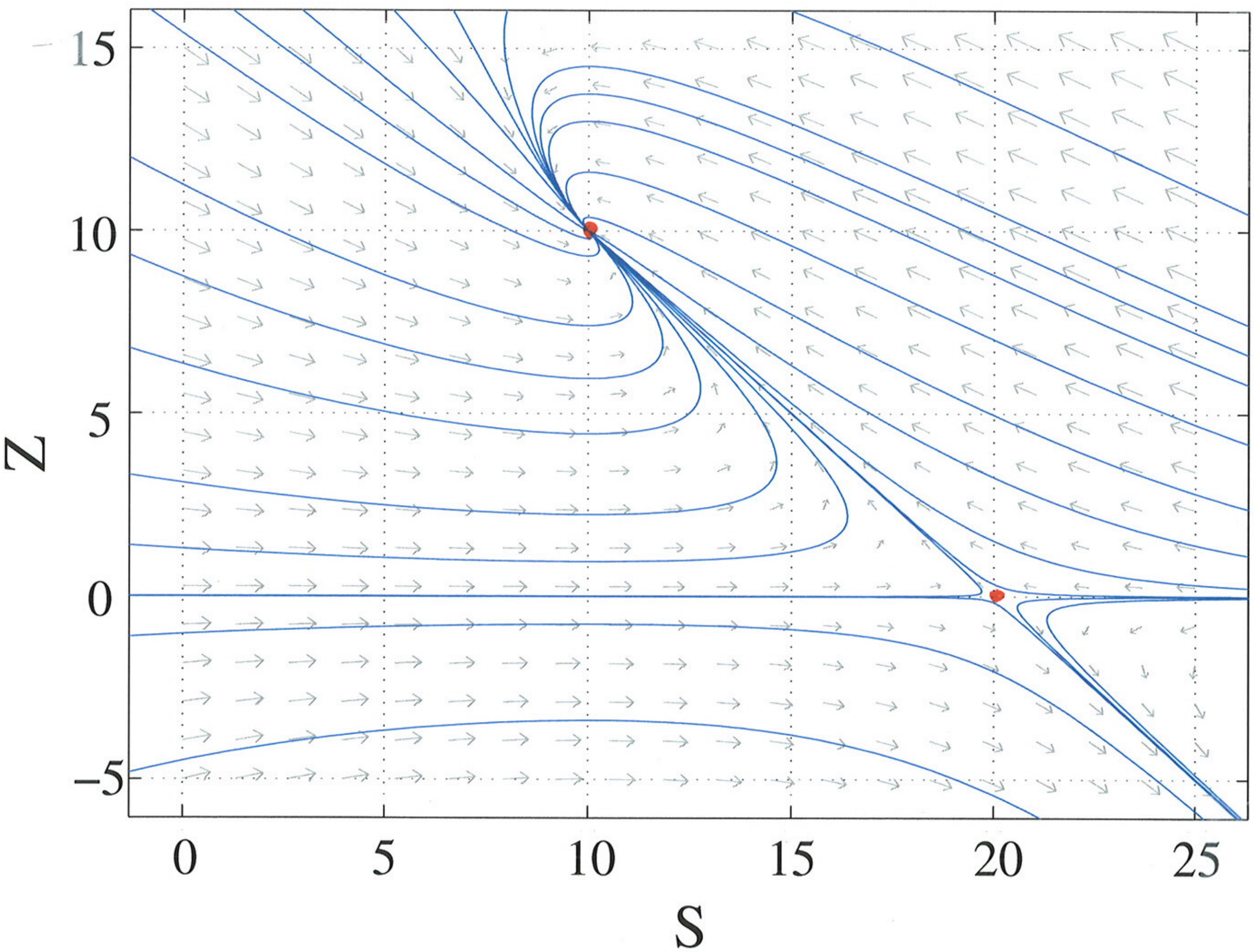
→ nonlinear term.

Interested in solution trajectories → phase plane.

(Λ = 20 ppl / day, r = 1 per day,

δ = 1 per day, β = 0.1 per contact per day)

Phase portrait:



Each trajectory (blue line) is a solution.

Notice:

- multiple critical points $\vec{x}_0 \neq 0$ s.t. $f(\vec{x}_0) = 0$ (in red).
  - $S = 20,\ Z = 0$ ← uninfected eq.
  - $S = 10,\ Z = 10$ ← endemic eq.

More careful definition of stability.

Given system $\vec{x}' = f(\vec{x})$ with solution $\vec{x}(t) = \vec{\phi}(t)$.

- A critical point $\vec{x}_0$ s.t. $\vec{f}(\vec{x}_0) = 0$ is STABLE if, given $\varepsilon > 0$, there is a $\delta > 0$ s.t. every solution $\vec{\phi}(t)$ that satisfies at $t=0$

$$\|\vec{\phi}(0) - \vec{x}_0\| < \delta$$ ← starts "close enough"

exists & satisfies for all $t \geq 0$

$$\|\vec{\phi}(t) - \vec{x}_0\| < \varepsilon.$$

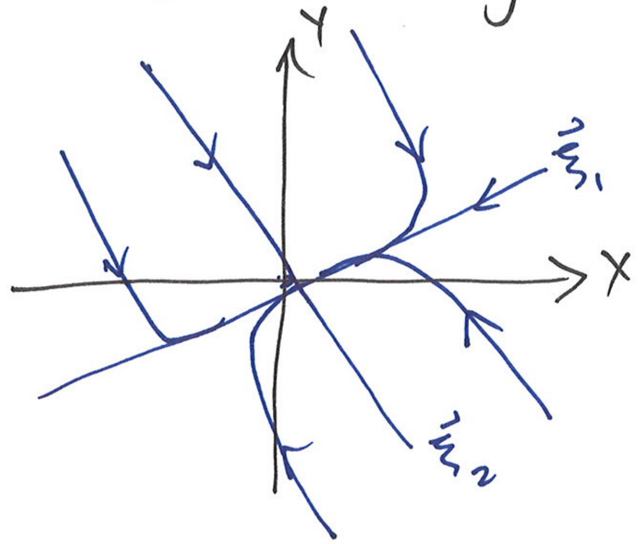
↝ solution bounded, stays around $\vec{x}_0$.

· A critical point $\vec{x}_0$ is asymptotically stable if it is stable **AND** there ~~is~~ exists some $\delta_0 > 0$ s.t. $\vec{x}(t) = \vec{\phi}(t)$ satisfies $\|\vec{\phi}(0) - \vec{x}_0\| < \delta_0$ "starts close enough"

then $\lim_{t \to \infty} \vec{\phi}(t) = \vec{x}_0$.

(~~our~~ def'n we had been using).

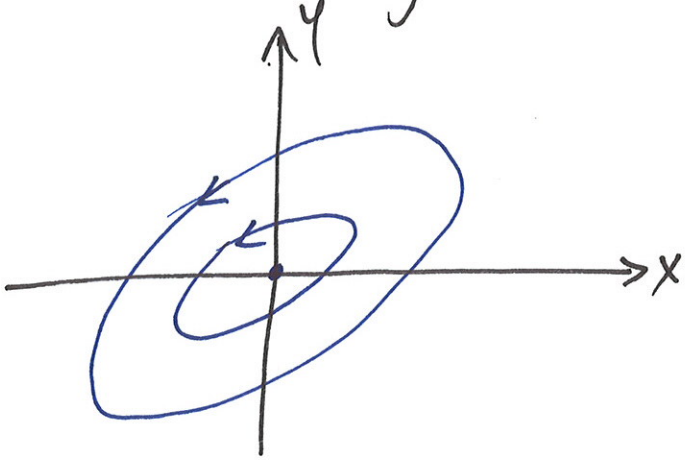
Eg] 2 negative, real eigenvalues (2×2 system) ($r_1 \neq r_2$)



$$\vec{x} = c_1 \xi_1 e^{r_1 t} + c_2 \xi_2 e^{r_2 t}.$$

here the origin is asymptotically stable.

Eg] 2 purely imaginary e-vals.
(2×2 system)



The origin is stable but not asymptotically stable.

Phase portraits of linear systems → review

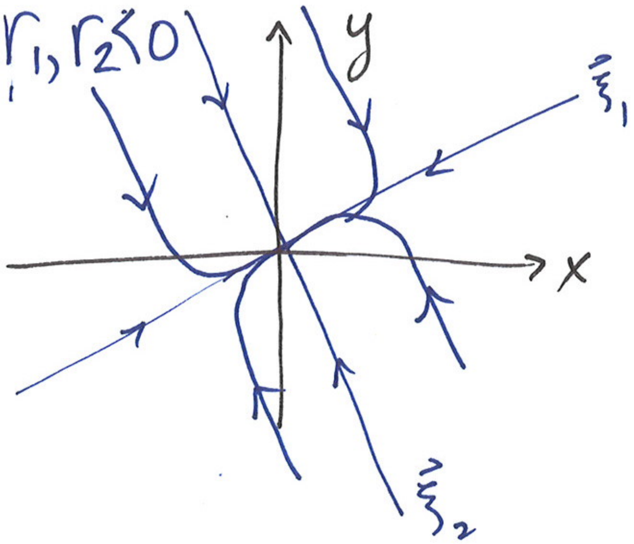
(Classify critical points).

CASE I: eigenvalues real, same sign.

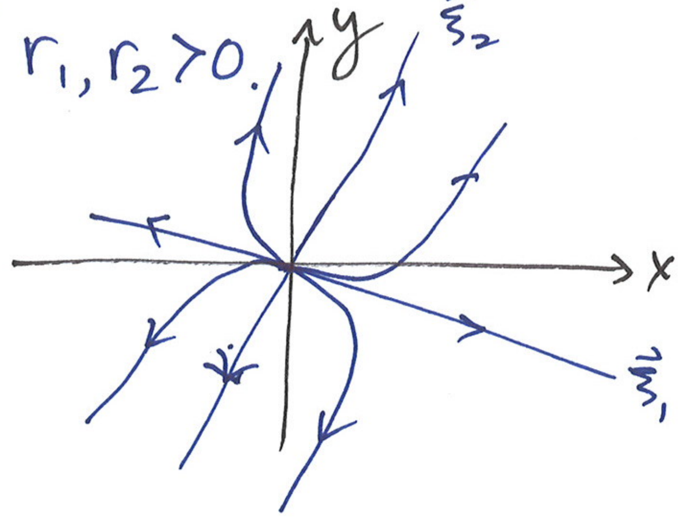
Sol: $\vec{x} = C_1 \vec{\xi}_1 e^{r_1 t} + C_2 \vec{\xi}_2 e^{r_2 t}$

e-val

e-vect.



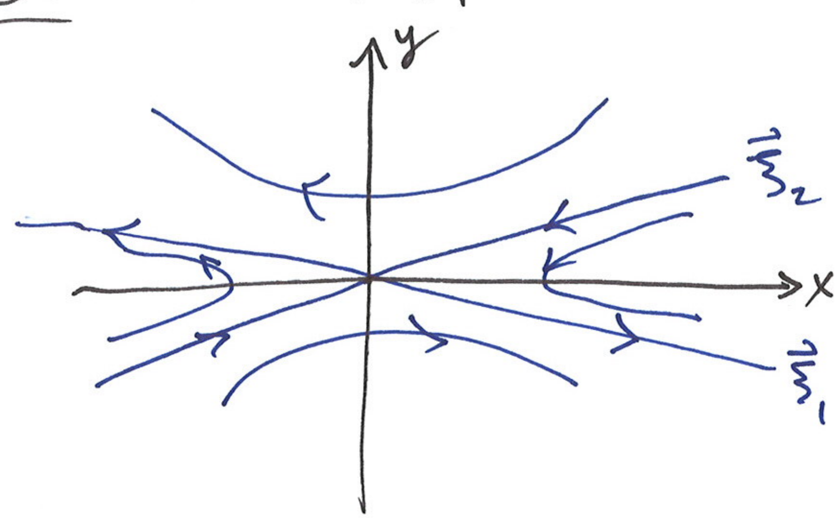
- stable, improper node
- book → "nodal sink"
- asymptotically stable.



- unstable, improper node
- book → "nodal source".
- unstable.

Case II: eigenvalues real + opposite sign ($r_1 > 0$, $r_2 < 0$).

Sol: $\vec{x} = C_1 \vec{\xi}_1 e^{r_1 t} + C_2 \vec{\xi}_2 e^{r_2 t}$.



- saddle point
- unstable.

- NEAR critical points, trajectories "look like" linear systems. trajectories.

  $S = 20$, $Z = 0$ → saddle point (1 +ve, 1 −ve).

  $S = 10$, $Z = 10$. → repeated e-val, <0, with 1 linearly indep. e-vect.

- In understanding systems, stability is key.

Therefore, our work in linear systems is a tool for understanding solution trajectories, critical point stability in nonlinear systems.

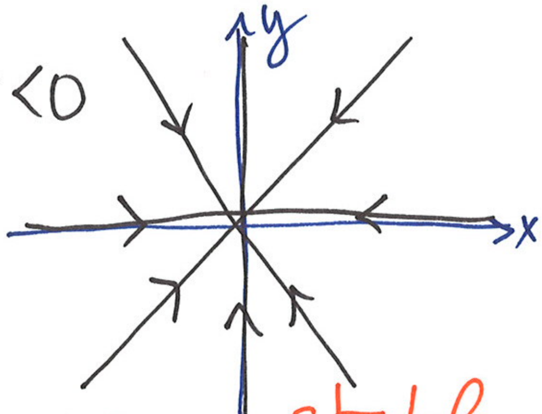
Case III: repeated real e-vals.

$r_1 = r_2 = r.$

(i) 2 lin. indep e-vects

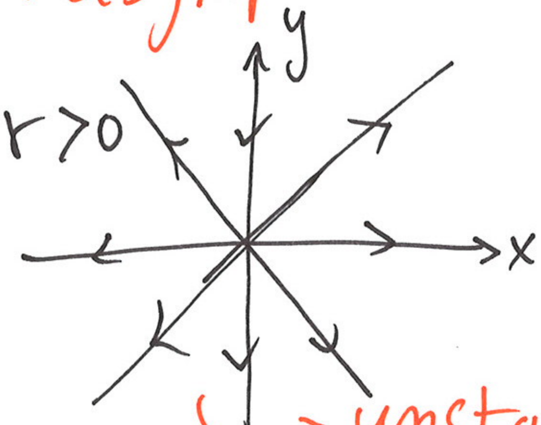
$$\vec{x} = C_1 \vec{\xi}_1 e^{rt} + C_2 \vec{\xi}_2 e^{rt}.$$

$r<0$



↳ asymp. stable

$r>0$

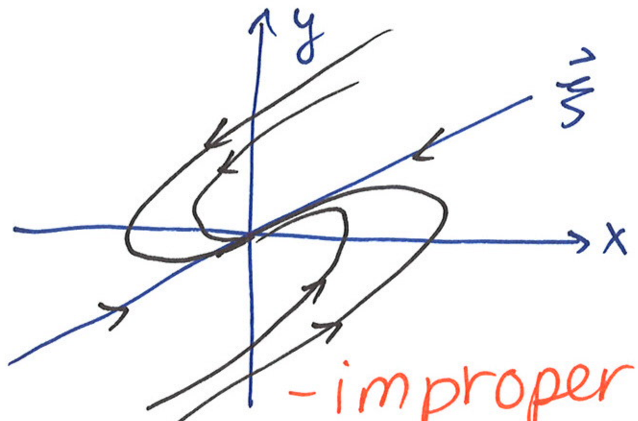


↳ unstable.

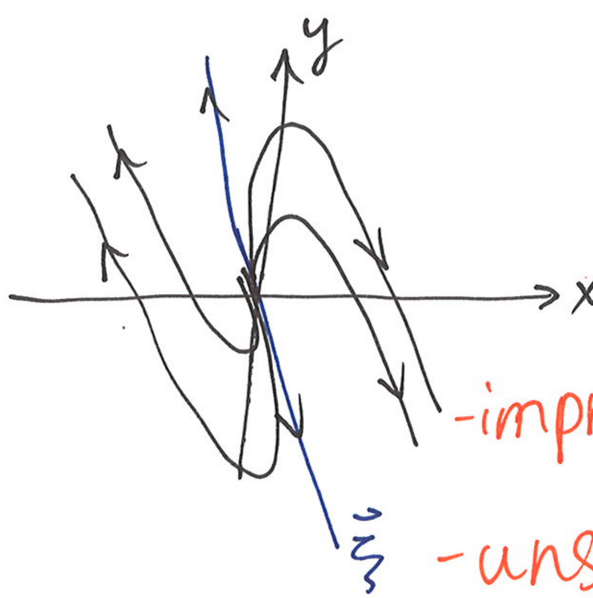
"star"
- proper node.
- ~~[illegible]~~

(ii) only 1 lin. indep. e-vect.

$$\vec{x} = C_1 \vec{\xi} e^{rt} + C_2 (\vec{\xi} t + \vec{\eta}) e^{rt}.$$



- improper node
- asymptotically stable.



- improper node
- unstable.