Announcements:

HW 10 → due Friday

HW 11 → due next Wed.

(both posted).

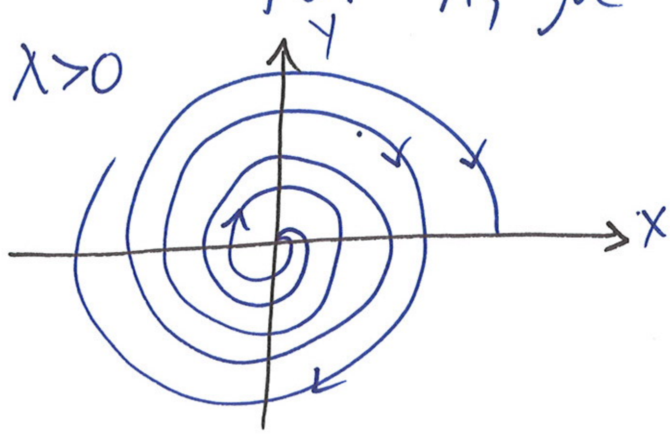
Survey for final review...

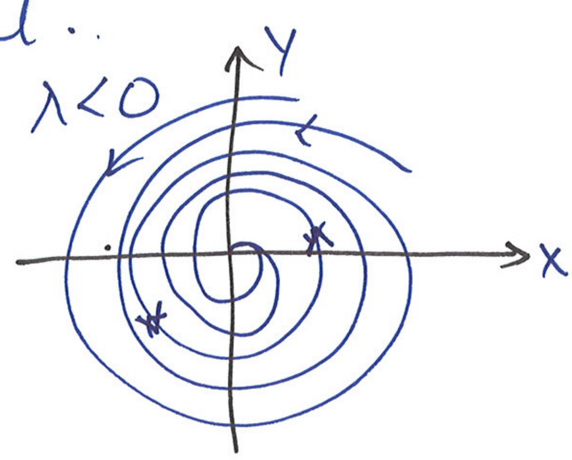
Phase Portraits

Case 4: Complex-valued eigenvals.

$r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$

for λ, μ real.





$r =$ distance from origin

$= \sqrt{x^2 + y^2}$

$\theta =$ angle from x-axis

$= \tan^{-1}(y/x).$

Consider

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

if $r^2 = x^2 + y^2$

<u>then</u> $\frac{d}{dt}(r^2) = \frac{d}{dt}(x^2 + y^2)$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \lambda r}$$

if $\tan\theta = (y/x)$

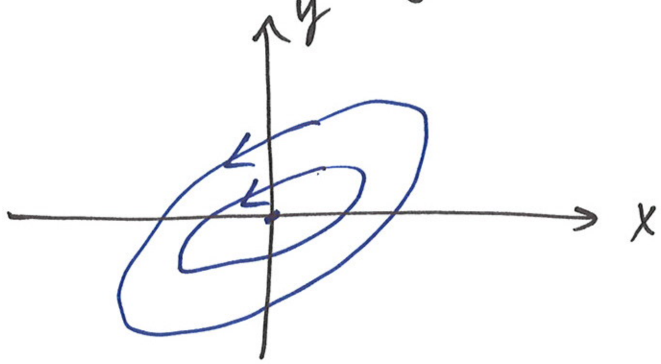
$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(y/x)$$

- spiral point
- unstable

- spiral point
- asymptically stable.

(Note: & spirals can be counter-clockwise, too)

$\lambda = 0$ (eigenvalues are purely imag)



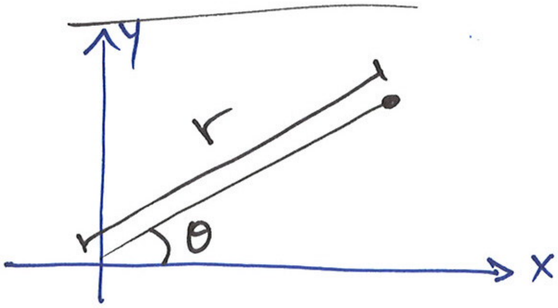
- center
- stable

(NOT asymp. stable)

(traj. can go clockwise, too).

(in all cases, critical point is origin, here).

Another way to think about this case:



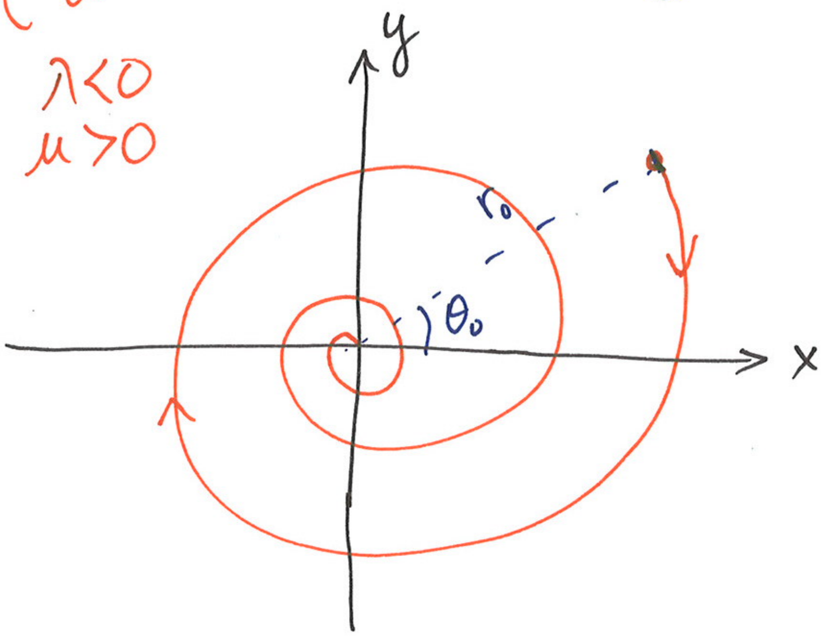
Use polar coords!

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{x}\frac{dy}{dt} - \frac{y}{x^2}\frac{dx}{dt}.$$

$$\Rightarrow \boxed{\frac{d\theta}{dt} = -\mu}$$

$$\begin{cases} \dfrac{dr}{dt} = \lambda r \\ \dfrac{d\theta}{dt} = -\mu \end{cases} \Rightarrow \begin{cases} r(t) = r_0 e^{\lambda t} \\ \theta(t) = -\mu t + \theta_0. \end{cases}$$

$\lambda < 0$
$\mu > 0$

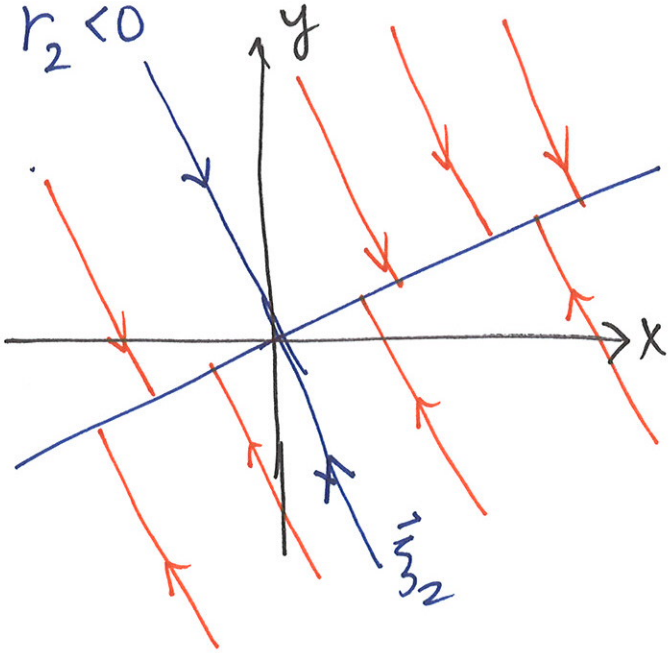


Given (r_0, θ_0) ... obtained the EXACT TRAJECTORY

(5) A zero eigenvalue!

$r_1 = 0, \quad r_2 \neq 0.$

<u>Solution</u>: $\vec{x} = C_1 \vec{\xi}_1 + C_2 \vec{\xi}_2 e^{r_2 t}.$



(if $r_2 > 0$, arrows outward)

"line of critical points".

stable $r_2 < 0$ (unstable $r_2 > 0$).

We've been (mostly) filling in phase planes with approximate trajectories.

Sometimes it's useful to know the EXACT trajectories. (especially in NL systems)

How? Phase plane Eq.

We have
$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

Then: $$\frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{g(x,y)}{f(x,y)}$$

$$\boxed{\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}}$$

function of x & y only.

phase plane eq

→ solve to get exact trajectories in the plane.

Note: Can only do this if system is autonomous.

Eg $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x \end{cases}$

Note: eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are ± 1 $\to$ saddle.
with corresponding e-vects $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for 1, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for -1.

Phase plane eq:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x}{y}.$$

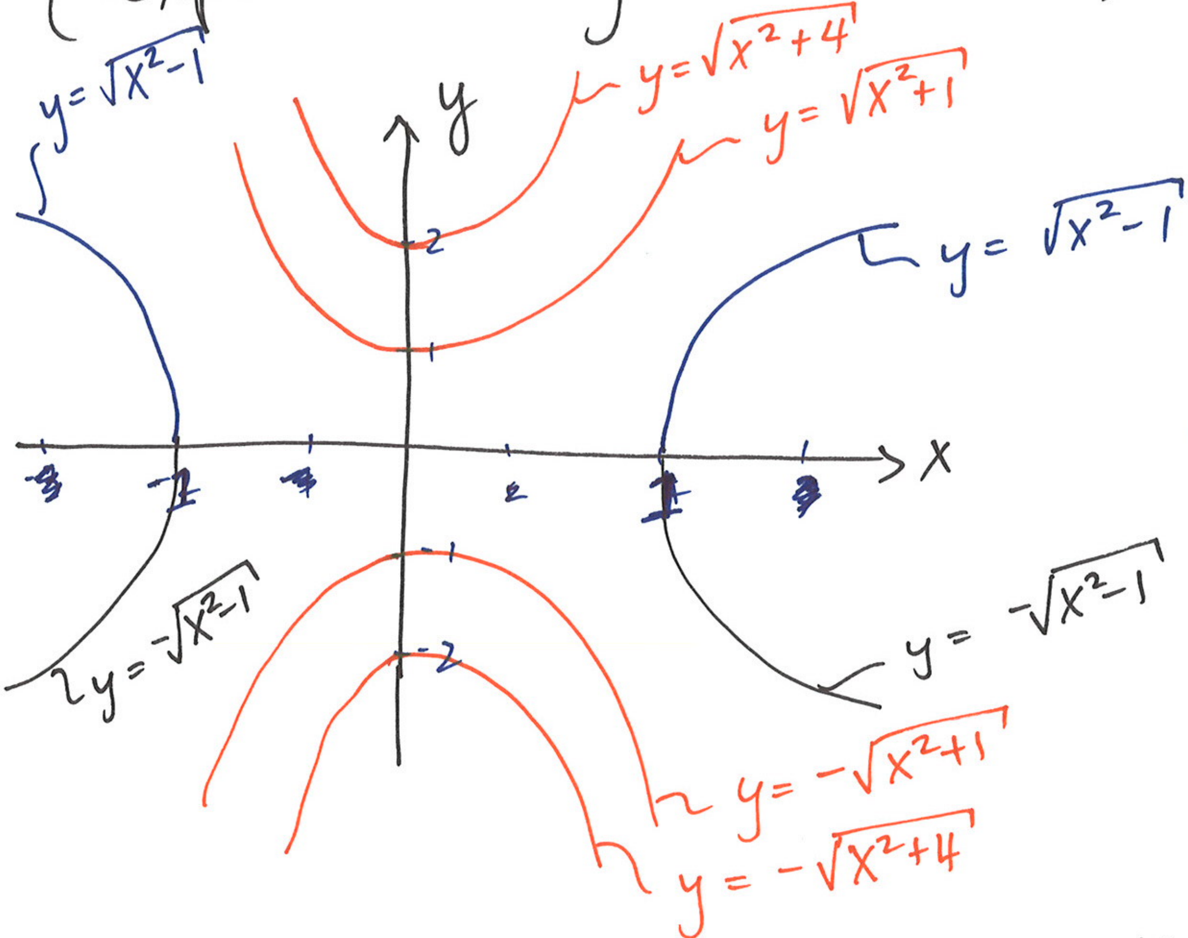
Separable! $\int y\,dy = \int x\,dx$

$$\frac{y^2}{2} = \frac{x^2}{2} + \tilde{C}$$

implicit. $\longrightarrow$ $\boxed{y^2 = x^2 + C.}$

all ~~traj~~ solution trajectories follow that equation.

(explicit $\rightarrow$ $y = \pm\sqrt{x^2 + c}$)



(not to scale)

recover saddle point behaviour as <u>expected</u>.