Homogeneous Equations

Def'n: For $\frac{dy}{dx} = f(x,y)$, if $f(x,y)$ can be expressed as a function of the ratio y/x only, then we say the equation is homogeneous

$$\frac{dy}{dx} = g(y/x).$$

Careful! "homogeneous" means different things in different contexts. We'll see others later.

Solve by: (1) pose substitution
(2) use sep. of vars.

Eg) $\frac{dy}{dx} = \frac{y+x}{x} = y/x + 1 = f(y/x).$

NOT separable! (could use an integrating factor)

To solve homogeneous eqs, pose substitution.

$$v = y/x.$$

Transform $\frac{dy}{dx} = \overbrace{\frac{y}{x} + 1}^{RHS}.$

RHS $= v+1.$ LHS?

LHS: $\frac{dy}{dx}$ but $y = v \cdot x$

$$= \frac{d}{dx}(vx) = v + x\frac{dv}{dx} \leftarrow \text{Product rule.}$$

~~Eq~~ Eq. becomes $v + x\frac{dv}{dx} = v + 1$

$$\frac{dv}{dx} = \underbrace{\frac{1}{x}}_{g(x)} \overbrace{(1)}^{p(v)}.$$

Solve by sep. of vars.

multiply by $dx/p(v)$ $\Rightarrow$ $dv = \frac{dx}{x}$

integrate both sides $\Rightarrow$ $\boxed{v = \ln(x) + c}$

Back in terms of y: $v = \frac{y}{x}$,

$$\frac{y}{x} = \ln(x) + c$$

$$\boxed{y = x\ln(x) + Cx}$$

<u>To solve homogeneous equations:</u> $\frac{dy}{dx} = f(y/x)$.

(1) Put RHS in terms of y/x only.

(2) Pose substitution $v = y/x$, $\left(\frac{dy}{dx} = v + x\frac{dv}{dx}.\right)$

$$\frac{dy}{dx} = f(y/x) \Rightarrow v + x\frac{dv}{dx} = f(v).$$

(3) Solve using separation of variables techniques.

- $$x\frac{dv}{dx} = f(v) - v$$

- $$\int \frac{dv}{f(v)-v} = \int \frac{1}{x}\,dx$$

- simplify.

(4) Put back interms of y & simplify.

Eg| $$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{y^2}$$

Check if homogeneous:

$$\frac{xy + y^2 + x^2}{y^2} \cdot \underbrace{\frac{1/x^2}{1/x^2}}_{=1}$$

$$= \frac{y/x + y^2/x^2 + 1}{y^2/x^2} = \frac{v + v^2 + 1}{v^2} \text{ after sub.}$$

Proceed through steps.

Integrating Factors (2.1 in text).

Want to solve eqs of the form

$$\frac{dy}{dx} + p(x) \cdot y = q(x) \qquad (1).$$

1st order, linear ODE.

We <u>do</u> know how to solve

$$\frac{dh}{dx} = r(x). \qquad (2).$$

(sep. of vars, etc).

Try to put (1) in same form as (2), which we can solve.

Let $h(x) = \underbrace{\mu(x)}_{\text{integrating factor.}} \cdot y(x).$

LHS of (2):

$$\frac{dh}{dx} = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx} \cdot y.$$

RHS of (2): $r(x)$.

$$\text{LHS} = \text{RHS}$$

$$\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx} y(x) = r(x).$$

$$\frac{dy}{dx} + \frac{1}{\mu}\frac{d\mu}{dx} y = \frac{r}{\mu}.$$

cf. $$\frac{dy}{dx} + p(x)\, y = q(x) \qquad (1).$$

- $\dfrac{r}{\mu} = q \quad \rightsquigarrow \quad \underline{r(x) = \mu(x) \cdot q(x)}$

- $\dfrac{1}{\mu}\dfrac{d\mu}{dx} = p(x).$ ODE for integrating factor!

$$\Rightarrow \int \frac{d\mu}{\mu} = \int p(x)\,dx$$

$$\ln|\mu| = \int^x p(x')\,dx'$$

$$\boxed{\mu = \exp\left\{\int^x p(x')\,dx'\right\}}$$

INTEGRATING FACTOR.

The integrating factor allows us to solve (1).

$$\frac{dy}{dx} + p(x)y = q(x).$$

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)q(x).$$

where $\mu(x) =$

Use sep. of vars to solve:

$$\int d[\mu(x)y(x)] = \int \mu(x)q(x).dx$$

$$\mu(x)y(x) = \int^x \mu(x')q(x')dx$$

$$\boxed{\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(x')q(x')dx' \\ \mu(x) &= e^{\int^x p(x')dx'} \end{aligned}}$$

STEPS: Eg| $\frac{1}{t}\frac{dy}{dt} + 3ty = t$.

① Put eq. in "Standard form": $\frac{dy}{dt} + p(t)y = q(t)$.
(coeffi. of 1st order derivative is 1).

eg| $\frac{dy}{dt} + 3t^2 y = t^2$.

② Compute IF $\mu(\cancel{t}) = e^{\int^t p(t')dt'}$

eg| $\mu(t) = e^{\int^t 3(t')^2 dt'} = e^{t^3}$.

③ Multiply eq. by $\mu(t)$ & reduce to integrable form.

$\rightarrow e^{t^3}\frac{dy}{dt} + 3t^2e^{t^3}y = t^2e^{t^3}.$

$h = \mu y$

$\frac{dh}{dx} = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y.$

$\rightarrow \frac{d}{dt}\left(e^{t^3}y\right) = t^2e^{t^3}$

4) Integrate + solve for y.

~~egx~~

$$\boxed{y(x) = \frac{1}{\mu}\int^{x}\mu(x')q(x')dx'}$$

eg) $\int d(e^{t^3}y) = \int t^2e^{t^3}dt$

$$e^{t^3}y = \frac{e^{t^3}}{3} + C$$

$$\boxed{y(t) = \frac{1}{3} + Ce^{-t^3}}$$

Eg1 Solve the IVP:

$$\begin{cases} \frac{1}{x}\frac{dy}{dx} - \frac{2}{x^2}y = x\cos x \\ y(\pi) = \pi^2 \end{cases}$$

Solve via an integrating factor.

① Standard form

$$\Rightarrow \frac{dy}{dx} \underbrace{- \frac{2}{x}}_{p(x)} y = \underbrace{x^2\cos x}_{q(x)}.$$

HAS to be there.

② Compute Integrating factor.

$$\mu(x) = \exp\left\{\int^x \left(\frac{-2}{x'}\right)dx'\right\} = e^{-2\ln(x)}$$

$$= e^{\ln(x^{-2})} = \boxed{x^{-2} = \mu(x)} = \left(\frac{1}{x^2}\right)$$

③ Multiply by μ & put in integrable form:

- $\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}\cancel{\frac{dy}{dx}}y = \cos x$
- $\frac{d}{dx}\Big(\underbrace{\frac{1}{x^2}y}_{\mu \cdot y}\Big) = \cos x$

4 Integrate + solve for y.

$$\int d\left(\frac{1}{x^2}y\right) = \int \cos x \, dx.$$

$$\frac{1}{x^2}y = \sin x + C$$

$$\boxed{y(x) = x^2 \sin x + Cx^2}$$ general solution.

Since $y(\pi) = \pi^2$

$$\Rightarrow C = 1$$

$$\boxed{y(x) = x^2 \sin x + x^2}$$ solution to IVP.