LECTURE 30

# Nonlinear Systems:

## The Plan:

→ Competing species model (9.4)

→ Cover concepts: critical point, linearization, almost linear systems, more phase portraits, basin of attraction. (9.2-9.3).

→ The simple pendulum next.

## Competing Species Model:

Classic Lotka-Volterra model.

Consider rabbits & sheep.

~~Encld~~

Include:

(1) Each species grows to some carrying capacity in absence of other.
→ logistic eq.
(Since rabbits are more prolific, higher growth)

(2) Competition: assume proportional to pop. sizes.
Assume conflicts reduce growth (more so for rabbits)

Let $x(t)$ = pop. of rabbits
$y(t)$ = pop. of sheep.

$$\frac{dx}{dt} = 3x\left(1 - \frac{x}{3}\right) - 2xy$$

growth rate 3

carrying capacity 3.

2x y competition

$$\frac{dy}{dt} = 2y\left(1 - \frac{y}{2}\right) - xy$$

growth rate 2

carrying capacity 2

less inhibited by comp.

Or
$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

$\boxed{\text{Model of rabbits \& sheep.}}$

First thing: Critical points.

Def'n: The points $\vec{x}_0$ where $f(\vec{x}_0) = \vec{0}$ are the critical (or equil. ~~of~~ or fixed) points of the autonomous system

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}).$$

To find critical points,
set $\dot{x}$ ~~A~~ $=0$ AND $\dot{y}=0$.

satisfy
$$x(3-x-2y)=0 \quad (1)$$
$$y(2-x-y)=0 \quad (2)$$

SIMULTANEOUSLY.

From (1): solve for $x$.

Either $x=0$ or $x=3-2y$

From (2): • if $x=0$, $y(2-y)=0$ (2)

so either $y=0$ or $y=2$.

• if $x=3-2y$, $y(2-(3-2y)-y)=0$ (2)

$$y(+y-1)=0$$

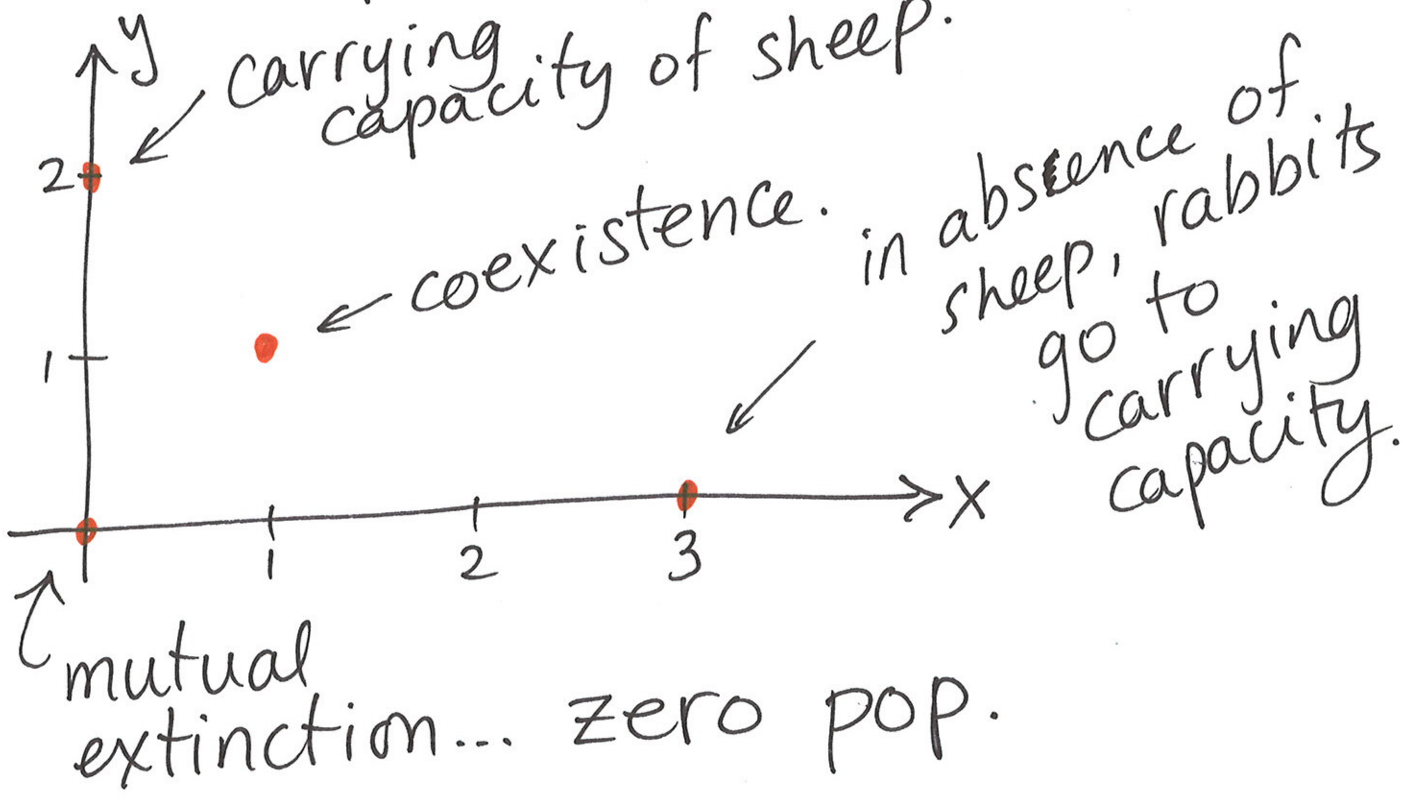
so either $y=0$ $(x=3)$

or $y=1$ $(x=1)$.

Therefore our critical points are:

$(0,0)$, $(0,2)$, $(3,0)$, $(1,1)$.

phase portrait:
(crit. points only)




To better understand solutions, need stability of crit. points, nearby dynamics.

Generically speaking,

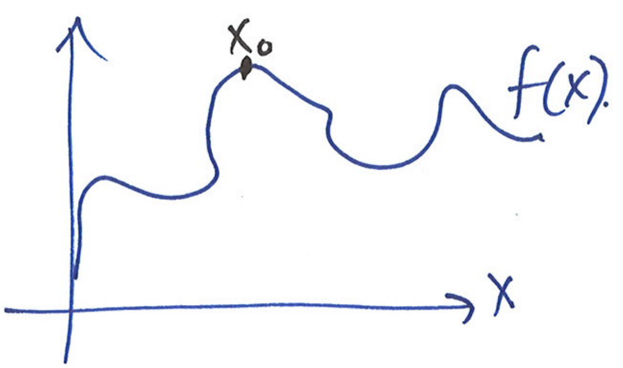
Want to investigate trajectories near & stability of critical points $\vec{x}_0$.

Consider: $\begin{cases} \dot{x} = F(x,y) \\ \dot{y} = G(x,y) \end{cases}$

with $x = x_0$, $y = y_0$ a crit. point.

Going to use the Jacobian.

Aside: recall Taylor series:




Taylor series

$$f(x) = f(x_0) + (x - x_0) f'(x_0)$$

$$+ \frac{(x-x_0)^2}{2} f''(x_0) + \cdots$$

higher-order terms.

close enough to $x_0$,

$$f(x) = f(x_0) + (x-x_0) f'(x_0)$$

is a good (linear) approx.

Multivariable T.S. expansions of $F(x,y)$, $G(x,y)$

(assume $F$, $G$ are at least twice continuously diff.)

$$F(x,y) = F(x_0,y_0) + (x-x_0)\frac{\partial F}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial F}{\partial y}(x_0,y_0) + \eta_1(x,y)$$

h.o.t.

$$G(x,y) = G(x_0,y_0) + (x-x_0)\frac{\partial G}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial G}{\partial y}(x_0,y_0) + \eta_2(x,y)$$

Write NL system as:

$J$:

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \boxed{\begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$+ \begin{pmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{pmatrix}.$$

centred about $(x_0, y_0)$

As $x \to x_0$, $y \to y_0$

$\eta_1, \eta_2 \to 0$ (h.o.t).

System is linearized.

$$\frac{d}{dt}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}\Bigg|_{x = x_0,\, y = y_0} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

like our matrix A before $\rightarrow$ Jacobian.

Could let $u = x - x_0$, ~~y~~ $v = y - y_0$.

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix}.$$

- type & stability of our critical points from eigenvalues/eigenvectors of ~~the~~ Jacobian $J(x_0, y_0)$ (evaluated at crit. point $(x_0, y_0)$).

"linearized system".

(matrix A from linear systems).

---

Works b/c F & G are twice ctsly diff → the system is "almost linear".

↳ idea: ~~linear~~ (there is a linear part) near critical point, nonlinear part is "small enough".

lin part

i.e. Suppose $\vec{x}' = A\vec{x} + g(\vec{x})$

$\vec{x}_0 = \vec{0}$ is our crit. point.

NL part.

system is "almost linear"

if $\dfrac{\|g(\vec{x})\|}{\|\vec{x}\|} \to 0$ as $\|\vec{x}\| \to 0$,

(in our work above,

$$\frac{\eta_1(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \to 0 \text{ as } (x,y) \to (x_0,y_0).)$$