Announcement:

Exam packages now for sale ... \$5 each

@ MATX 1119, Mon-Fri, 11-3.
(undergrad math club)

"

Given a nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

- Find the critical points (x_0, y_0)
s.t. $F(x_0,y_0) = G(x_0,y_0) = 0$

(i.e. where $\frac{dx}{dt} = \frac{dy}{dt} = 0$)

(found for rabbits & sheep example).

- Characterize those critical points. (type & stability of points, solution trajectories "near" points) by linearizing about the critical points and examining the resulting linear system.

Let $x(t) = x_0 + \tilde{x}(t)$
$y(t) = y_0 + \tilde{y}(t)$

($\tilde{x}(t)$, $\tilde{y}(t)$ "small")

Then

$$\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \underbrace{\begin{pmatrix}F_x(x_0,y_0) & F_y(x_0,y_0)\\ G_x(x_0,y_0) & G_y(x_0,y_0)\end{pmatrix}}_{\text{Jacobian at }(x_0,y_0)}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$$

(for F & G "nice" → Wed).

Examine this linearized system.

Back to rabbits & sheep.

$$\dot{x} = x(3-x-2y) = F(x,y)$$

$$\dot{y} = y(2-y-x) = G(x,y)$$

with critical points

$(0,0)$, $(0,2)$, $(3,0)$, $(1,1)$.

For each critical point (x_0, y_0).

can let $x = x_0 + \tilde{x}$ $(\tilde{x}, \tilde{y}$ small$)$.

$y = y_0 + \tilde{y}$

distance from critical point.

Linearizing,

$$\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0)\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}.$$

Well...

$F_x(x,y) = 3-2x-2y$

$F_y(x,y) = -2x$

$G_x(x,y) = -y$

$G_y(x,y) = 2-2y-x$.

$J(x_0, y_0)$.

$$\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix} 3-2x_0-2y_0 & -2x_0 \\ -y_0 & 2-2y_0-x_0\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$$

in general.

Now consider each fixed point.

→ determine stability

→ draw pieces of phase portrait.

Start with origin! $(0,0)$

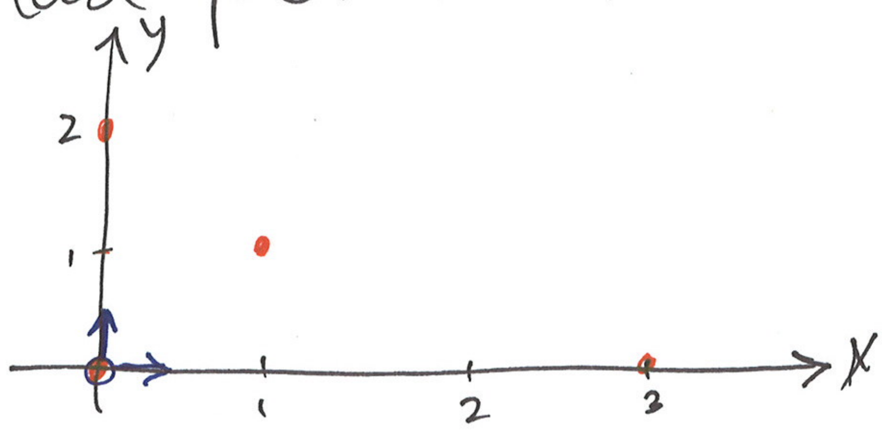
$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

eigenvalues $r_1 = 3$, $r_2 = 2$

corres. e-vects. $\vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

e-vals both real, positive → UNSTABLE.

Phase portrait.



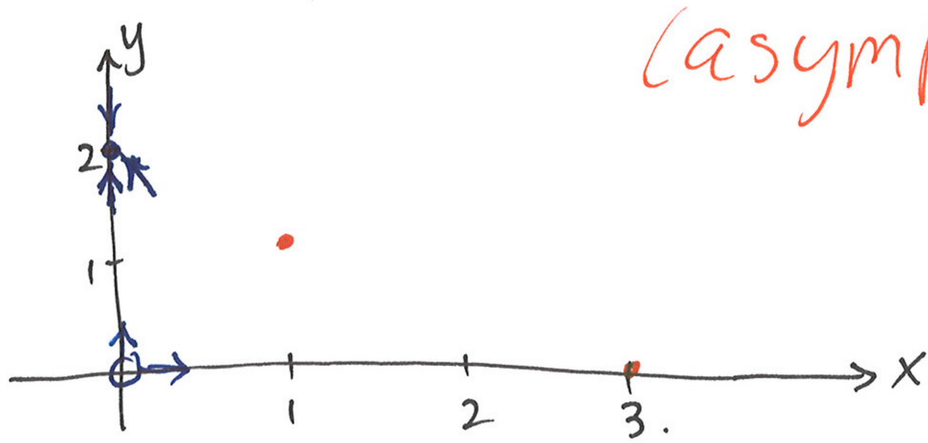
Now for (0, 2) → sheep only.

$$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

eigenvalues $r_1 = -1$, $r_2 = -2$

w/ e-vects $\vec{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

real, -ve e-vals
→ stable node (sink).
(asymp)



Now for (3, 0) → rabbits only.

$$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

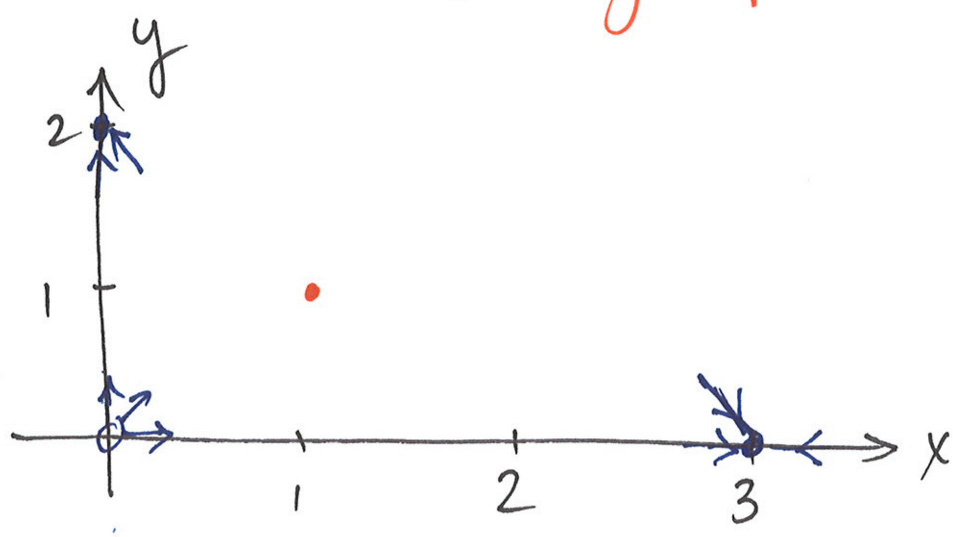
eigenvalues $\quad r_1 = -3 \quad , \quad r_2 = -1$

w/ e-vects $\quad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \vec{\xi}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$

real, -ve. e-vals

$\rightarrow$ stable node (sink).

(asymp.)

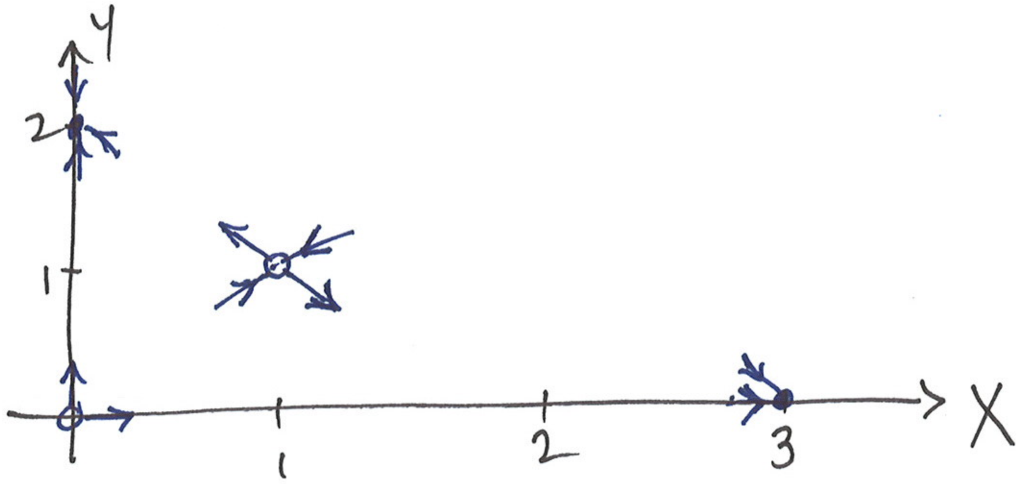


Now for $(1,1)$ $\rightarrow$ coexistence.

$$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

e-vals $\quad r_1 = -1+\sqrt{2} \quad r_2 = -1-\sqrt{2}$

w/ e-vects $\quad \vec{\xi}_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \quad \vec{\xi}_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$



eigenvals real, opposite sign
→ saddle point (unstable)

Now we can use common sense to fill in approximate trajectories.
(or use phase plane eq. to get exact ones).

One more tool →
<u>Nullclines</u>

lines along which

$\dot{x} = 0$ (vertical directions) only

OR $\dot{y} = 0$ (horizontal directions only).

in ~~d~~ direction field.

(intersect at critical points).

Here: • $\dot{x} = x(3 - x - 2y)$

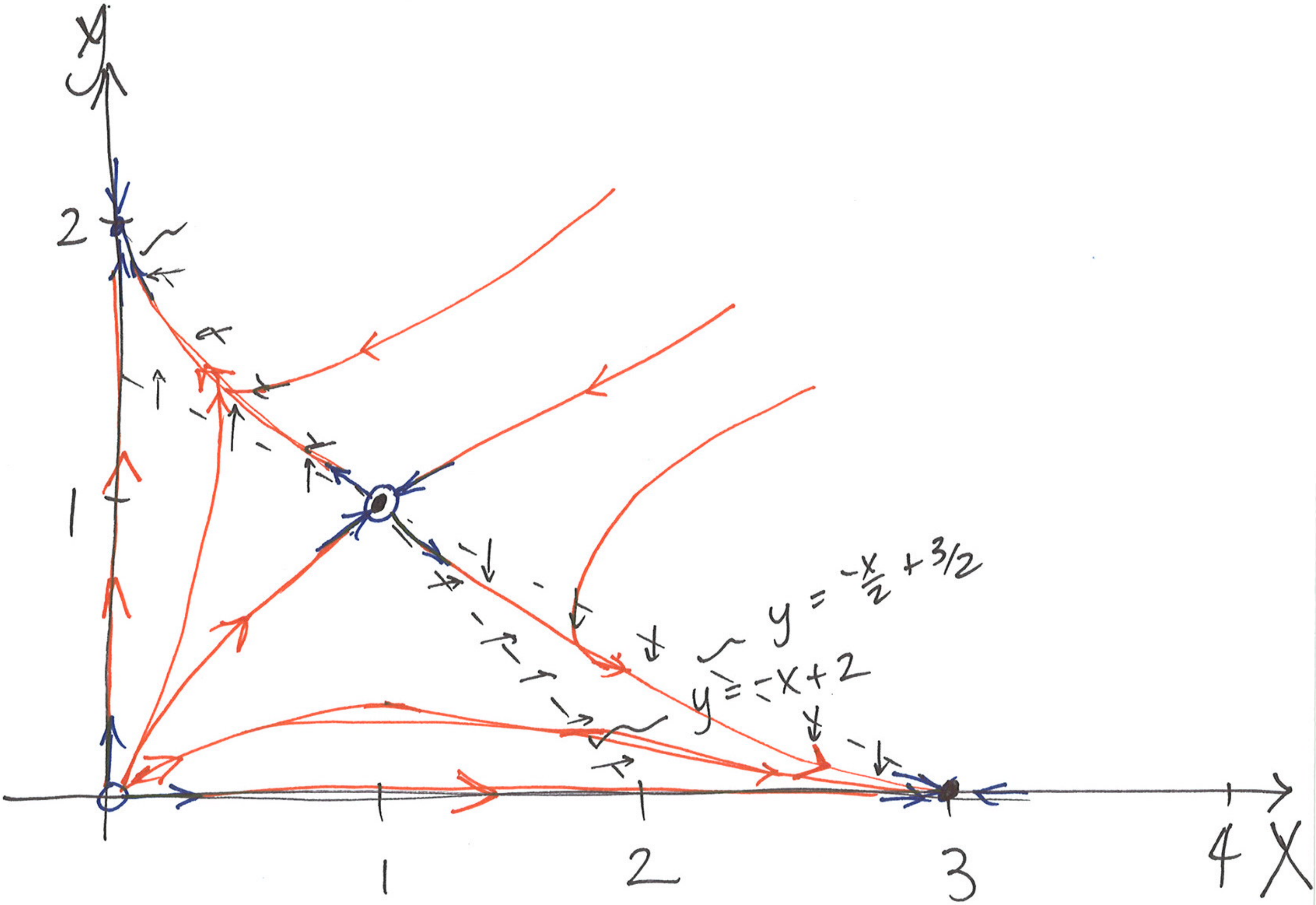
→ vertical nullclines along $x = 0$ and $y = \frac{-x}{2} + \frac{3}{2}$

(get dir. from $\dot{y}$)

• $\dot{y} = y(2 - x - y)$

→ horizontal nullclines along $y = 0$ & $y = -x + 2$. (direction from $\dot{x}$)

Phase portrait



Check dir on nullcline $y = \frac{-x}{2} + \frac{3}{2}$.

$(x = 3 - 2y)$

$$\dot{y} = y(2-(3-2y)-y)$$
$$= y(-1+y)$$

so $\dot{y} > 0$ for $y > 0$ & $y > 1$ (up)

$\dot{y} < 0$ for $y > 0$ & $y < 1$ (down)