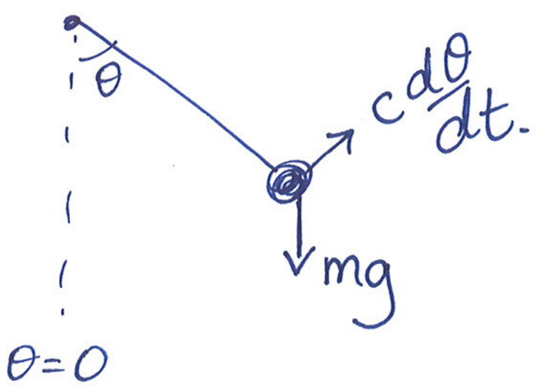
Undamped, Nonlinear Pendulum:



($c = 0$ for now)
(undamped)

Equation:

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \qquad (\omega^2 = g/L)$$

As a system:

Let $\theta = x$, $y = \dot{\theta}$:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 \sin x \end{cases}$$

system for NL pend.

Critical points where

$\dot{x} = 0$ AND $\dot{y} = 0$.

ang. velocity

angular accel.

Are: $y = 0$ (angular vel. $= 0$)

AND. $x = n\pi$ for $n \in \mathbb{Z}$.

n even $\rightarrow$ down pos.
($n = 2m$).

$m \in \mathbb{Z}$

n odd $\rightarrow$ up pos
($n = 2m+1$)

Anticipate:

- stable ~~so~~ centre for n even.
- Saddle for n odd.

Compute:

First find Jacobian J:

$$J(x,y) = \begin{pmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{pmatrix} \quad \text{for} \quad \begin{matrix} \dot{x} = F(x,y) \\ \dot{y} = G(x,y) \end{matrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{pmatrix}, \text{ here.}$$

Linearize about down equil. $(x = 2m\pi,\ y = 0)$

$\hookrightarrow x = 0.$

Let $x = 2m\pi + \tilde{x}$ $\quad(\tilde{x}, \tilde{y}$ "small").

$y = 0 + \tilde{y}$

crit. point.

Then $\cancel{\frac{d\tilde{x}}{dt} =}$ $\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = J(2m\pi, 0)\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

Type and stability of critical

point from e-vals/e-vects of $J(2m\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$.

eigenvalues are $\pm i\omega$

corresp. eigenvects $\begin{pmatrix} 1 \\ \pm i\omega \end{pmatrix}$.

Since e-vals are purely imaginary, $(2m\pi, 0)$ (or $(0,0)$, $(2\pi,0)$, etc) ~~is~~ are stable centres

$(m \in \mathbb{Z})$.

Linearize about up equil.

$(x = (2m+1)\pi,\ y = 0$

$\hookrightarrow x = -\pi, \pi, 3\pi, \ldots$

$x = (2m+1)\pi + \tilde{x}$ $\quad (\tilde{x}, \tilde{y}$ small)

$y = 0 + \tilde{y}$

Obtain $\frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = J((2m+1)\pi, 0)\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}.$

Jacobian evaluated at crit. point.

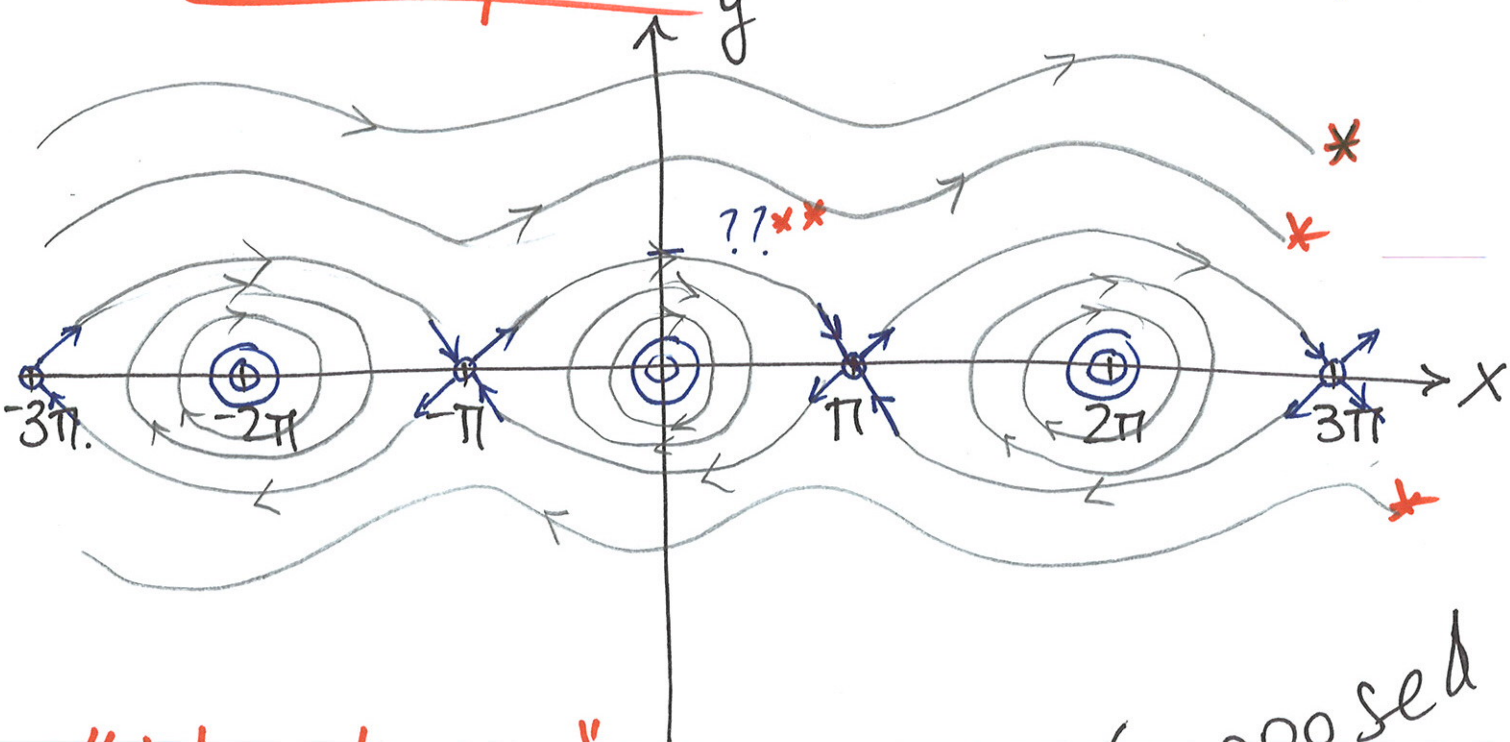
- type & stab of crit. point from e-vals/e-vects of $J((2m+1)\pi, 0)$.

$$J((2m+1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix}.$$

eigenvalues are $r = \pm\omega$
with eigenvects $\begin{pmatrix} 1 \\ \pm\omega \end{pmatrix}$.

e-vals are real & opposite sign $\rightsquigarrow$ saddle point.

Phase portrait: $(\omega = 1)$



(supposed to look the same)

* "librations"

initial vel. so high pendulum goes over the top.

** use conservation of energy to show this value $\cancel{\theta_{max}} = \dot{\theta}_{max} = 2\omega$.
$(= y)$.

Powerful theory! We used linearization to get an approx. picture of the full NL system.

Compare with Linear (simple) pendulum
→ show approximation.

PERIOD of oscillations.

Linear pendulum

$$\ddot{\theta} + \omega^2 \theta = 0$$

Period was $\frac{2\pi}{\omega}$ $\left(= 2\pi\sqrt{\frac{L}{g}}\right)$.

NL pendulum:

conservation form of ODE:

$$\frac{\dot{\theta}^2}{2} - \omega^2 \cos\theta = C$$

Assume you drop pendulum ($\dot{\theta}(0) = 0$) at angle θ_0.

$$\Rightarrow C = -\omega^2 \cos\theta_0$$

$$\frac{\dot{\theta}^2}{2} - \omega^2 \cos\theta = -\omega^2 \cos\theta_0$$

(for pend. dropped at θ_0)

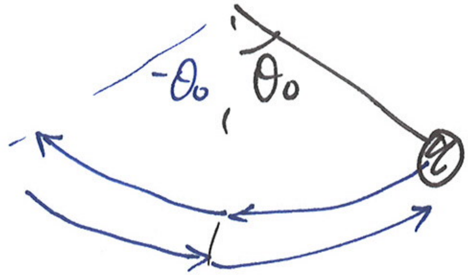
Solve for $\dot{\theta}$:

$$\frac{d\theta}{dt} = -\sqrt{2w^2(\cos\theta - \cos\theta_0)}$$

← - b/c towards equil.

$$dt = \frac{-d\theta}{\sqrt{2w^2(\cos\theta - \cos\theta_0)}}$$

Period



$$\int_0^{T/4} dt = \frac{-1}{\sqrt{2w^2}} \int_{\theta_0}^{0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

$$T = \frac{+4}{\sqrt{2w^2}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

(can write as elliptic int.)

period depends on amplitude $\underline{\underline{\theta_0}}$. (different)

$\Rightarrow$ truly are approximating BUT phase portrait is <u>very</u> helpful.