$\boxed{\text{Lecture 4}}$

Existence & Uniqueness

(Section 2.4 of text; more details in 2.8)

Eg $\begin{cases} x\dfrac{dy}{dx} - 3y = x^5\cos x \\ y(0) = 1 \end{cases}$ (Motivating Example)

Use an ~~in~~ IF.

① Put in "standard form"

$$\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{p(x)} y = x^4 \cos x.$$

② Compute IF: $\mu(x) = e^{\int^x p(x')\,dx'} = e^{\int^x -\frac{3}{x'}\,dx'}$

$$= e^{-3\ln x} = \boxed{\frac{1}{x^3} = \mu(x)}$$

③ Multiply by μ & put in integrable form:

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = x\cos x$$

$$\frac{d}{dx}\Big[\underbrace{\frac{1}{x^3}}_{\mu(x)}y\Big] = x\cos x$$

④ Integrate & solve for y:

$$\int \frac{d}{dx}\left[\frac{1}{x^3}y\right]dx = \int x\cos x\,dx$$

integrate RHS by parts.

$u = x \quad dv = \cos x\,dx$

$du = dx \quad v = \sin x$

$= \cancel{\int} u\,dv - \int v\,du.$

$$\frac{1}{x^3} y = \cos x + x \sin x + C$$

$$\boxed{y(x) = x^3 \cos x + x^4 \sin x + Cx^3}$$ general solution

Use initial condition to get solution of IVP.

$$1 = y(0)$$
$$= (0)^3 \cos 0 + (0^4) \sin 0 + C(0)^3$$
$$1 = 0 \text{ ?!? }$$ doesn't make sense.

Existence & uniqueness theorem tells us what's going on.

Useful → does a DE even <u>have</u> a solution? (before spending hours solving it).

→ Once a solution is obtained... are there others?

Theorem: If the functions p and g are continuous on an open interval $\alpha < t < \beta$ containing the point $t = t_0$, then there exists a <u>unique</u> function $y = \phi(t)$ that satisfies the ~~IVP~~ DE

$$\frac{dy}{dt} + p(t)y = g(t)$$

for each t in $\alpha < t < \beta$, and also satisfies

$$y(t_0) = y_0$$

where y_0 is a prescribed initial value.
(2.4.1 in text)

Proof: see derivation of integrating factor!
in the end we had

$$y = \frac{1}{\mu(t)} \int^t \mu(t') g(t') dt' \ldots$$

When we did this, we assumed (without statement) that p & g were cts & integrable.

What this means for you:

for all 1st order, linear equations where p, g cts in range containing initial point, a unique solution exists.

Eg) Find an interval over which the IVP

$$\begin{cases} x\frac{dy}{dx} - 3y = x^5 \cos x \\ y(\pi) = 0 \end{cases}$$

has a unique solution.

Standard form: $\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{p(x)} y = \underbrace{x^4 \cos x}_{g(x)}.$

Check where $p(x)$ & $g(x)$ are continuous:

$g(x) = x^5 \cos x \Rightarrow$ cts everywhere.

$p(x) = \frac{-3}{x} \Rightarrow$ cts on $-\infty < x < 0$ (1)

& $0 < x < \infty$. (2)

IVP with IC $y(\pi) = 0 \Rightarrow$ initial value of x is π, $0 < \pi < \infty \rightarrow$ in interval (2).

$\therefore$ The IVP given has a unique solution on $0 < x < \infty$ (aka $x > 0$).

What if IC was $y(-\pi) = 0$?

then unique solution on $-\infty < x < 0$

aka $x < 0$.

Motivating problem:

IC $y(0) = 1$.

- 0 not in interval (p not cts at $x=0$)
- theorem assumptions don't hold.
- solution does not exist.

Think about at home:

IC $y(0) = 0$?

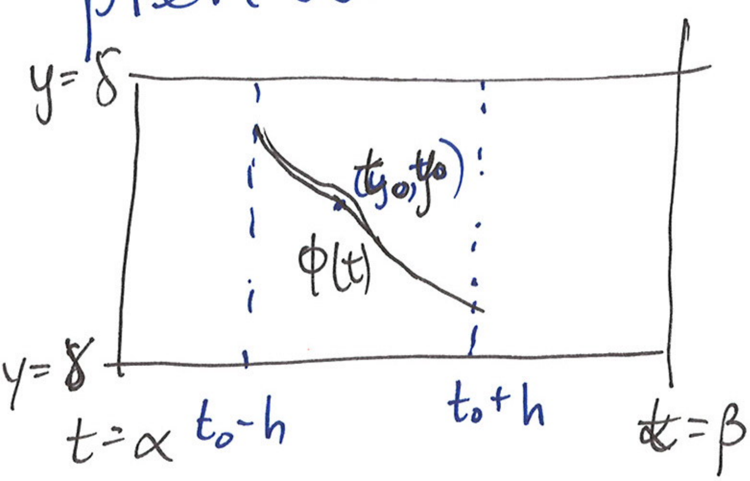
- theorem assumptions violated
- solution not unique.

That's for linear eqs. For nonlinear eqs we have a more general theorem:

Theorem: Let functions f and $\partial f/\partial y$ be cts in $\gamma < y < \delta$, $\alpha < t < \beta$ containing (y_0, t_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution to the IVP $(y = \phi(t))$.

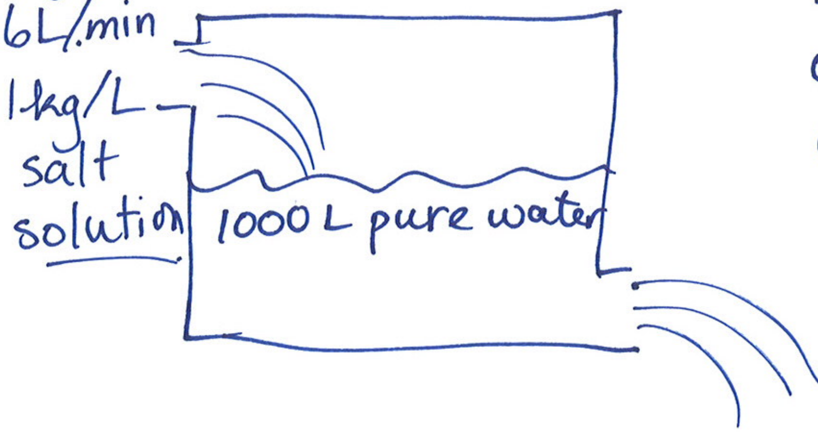
$$\begin{cases} \dfrac{dy}{dt} = f(y,t) \\ y(t_0) = y_0 \end{cases}$$

(if $f(y,t) = -p(t)y + g(t)$, reduces to previous theorem).



Applications - Models with 1st order eqs (2.3 of text).

Eg Mixing.



When will concentration of salt in tank be 0.5 kg/L?

Think of tank as compartment containing salt.

$\longrightarrow X(t) = \#$ kg at time t.

(Note: volume of fluid in tank is const. 100L)

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}.$$

$$\longrightarrow \boxed{X(t)} \longrightarrow$$

$$\text{rate in} = \underbrace{\left(\frac{1\,kg}{L}\right)}_{\text{Concentration}} \cdot \underbrace{\left(\frac{6L}{min}\right)}_{\text{Vol. in}} = 6\,kg/min.$$

$$\text{rate out} = \left(\begin{matrix}\text{conc.}\\ \text{at } t\end{matrix}\right) \cdot \left(\begin{matrix}\text{vol}\\ \text{out}\end{matrix}\right)$$

$$= \frac{x(t)}{1000L} \cdot \frac{6L}{min}$$

rate out $= \frac{3x}{500}$. kg /min.

$$\begin{cases} \frac{dx}{dt} = 6 - \frac{3x}{500} \\ x(0) = 0 \end{cases}$$

IVP for mass of salt.