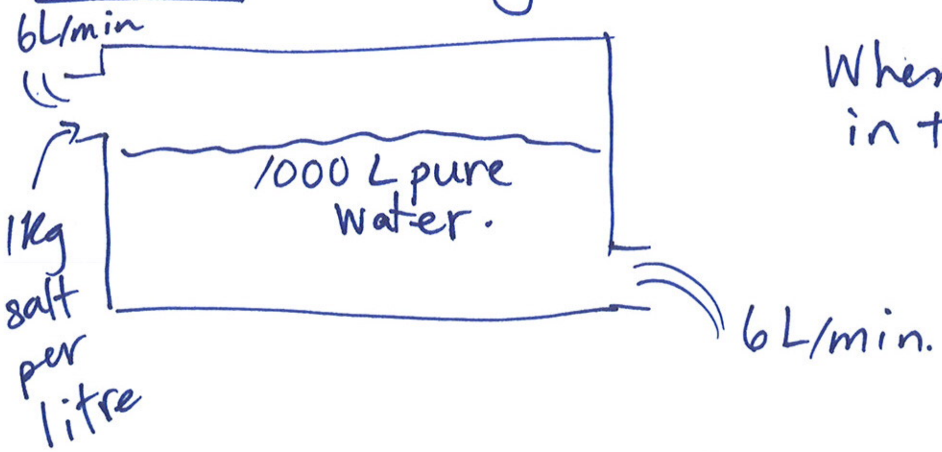
Last time: Mixing Problem.



When will the concentration in the tank be 0.5 kg/L?

Let $x(t)$ = mass of salt at time t.

Then we found
$$\begin{cases} \dfrac{dx}{dt} = 6 - \dfrac{3x}{500} \\ x(0) = 0 \end{cases}$$
IVP for mass of salt.

How would you solve this?

(1) sep. of vars.

(2) integrating factor.

Solve to obtain mass of salt at time t:
$$x(t) = 1000(1 - e^{-(3/500)t}) \text{ kg}.$$

Volume in tank constant 1000L

Concentration $C(t) = \dfrac{x(t) \text{ kg}}{1000 \text{ L}} = \left(1 - e^{\frac{-3t}{500}}\right) \text{kg/L}.$

Desired quantity: time τ when $C(\tau) = 0.5$ kg/L.
$$1 - e^{-3\tau/500} = 0.5$$
$$e^{-3\tau/500} = 0.5$$
$$\ln(\qquad) = \ln(\quad)$$

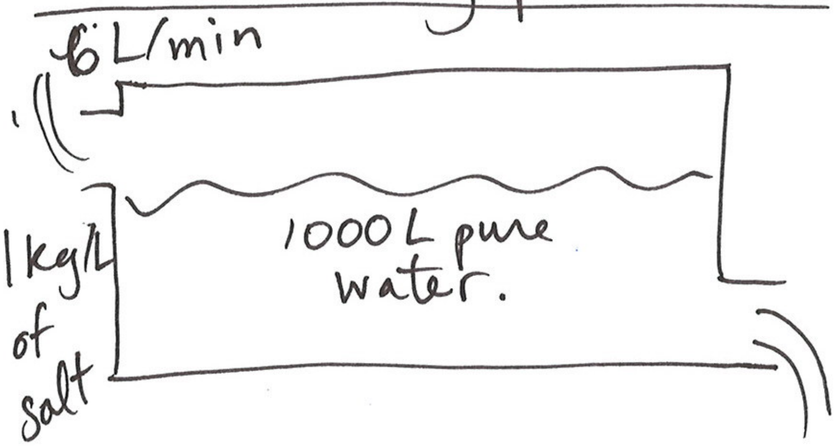
$$-\frac{3\tau}{500} = \ln\left(\frac{1}{2}\right) = -\ln(2).$$

$$\boxed{\tau = \frac{500\ln(2)}{3} \text{ min}}$$

$$\boxed{\approx 115.52 \text{ min}}$$

The concentration of salt in the tank is 0.5 kg/L at $\frac{500\ln(2)}{3} \approx 115.52$ min.

Another mixing problem.



Find concentration at time t.

Let $x(t)$ be the mass at time t.

This time volume not constant! Let $V(t)$ be ~~mass~~ volume at time t.

Determine volume first.

$$\frac{dV}{dt} = \text{rate in} - \text{rate out}$$
$$= 6\,L/\text{min} - 5\,L/\text{min}$$

$$\frac{dV}{dt} = 1 \; ; \quad V(0) = 1000\,L.$$

$$\begin{cases} \dfrac{dV}{dt} = 1 \\ V(0) = 1000 \end{cases} \Rightarrow \boxed{V(t) = 1000 + t.}$$

For mass of salt:

$\dfrac{dx}{dt}$ = rate in − rate out.

rate in = (concentration in) (volume in)

$1\,\text{kg/L} \cdot 6\,\text{L/min}$

$= 6\,\text{kg/min}.$

units for $\dfrac{dx}{dt}$ are $\dfrac{\text{mass}}{\text{time}}$ (x is mass) (t is time)

all terms must have same units

rate out = (concentration out) (volume out)

$$= \left(\frac{x(t)}{1000+t}\right) 5\,\text{L/min} = \frac{5x}{1000+t}.$$

$$\begin{cases} \dfrac{dx}{dt} = 6 - \dfrac{5x}{1000+t} \\ x(0) = 0 \end{cases}$$ IVP for mass.

Solve using integrating factor

① Eq. in standard form

$$\frac{dx}{dt} + \underbrace{\left(\frac{5}{1000+t}\right)}_{\text{"}p(t)\text{"}} x = \underbrace{6.}_{\text{"}q(t)\text{"}}$$

② Compute IF

$$\mu(t) = \exp\int^{t} p(t')\,dt' = e^{\int \frac{5}{1000+t}\,dt}$$

$$= e^{5\ln(1000+t)} = e^{\ln[(1000+t)^5]}$$

$$\boxed{\mu(t) = (1000+t)^5}$$

③ Multiply eq. by $\mu(t)$ + put in integrable form.

$$(1000+t)^5 \frac{dx}{dt} + 5(1000+t)^4 x = 6(1000+t)^5$$

$$\frac{d}{dt}\left[(1000+t)^5 x\right] = 6(1000+t)^5$$

④ integrate:

$$(1000+t)^5 x = \int^{t} 6(1000+t')^5\,dt'$$

$$= (1000+t)^6 + C$$

$$\boxed{x(t) = 1000+t + \frac{C}{(1000+t)^5}}$$

Find constant from initial condition:

$$X(0) = \cancel{1000}0 \Rightarrow C = -1000^6$$

$$\boxed{X(t) = 1000 + t - \frac{1000^6}{(1000+t)^5} \text{ is mass at time } t.}$$

What's concentration at time t?

$$C(t) = \frac{X(t)}{V(t)} = \frac{X(t)}{1000+t}.$$

$$\boxed{C(t) = 1 - \frac{1000^6}{(1000+t)^6}}$$

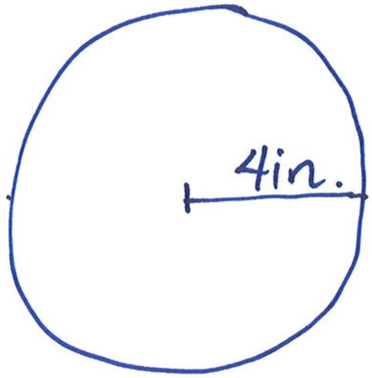
Does this make sense? Yes.

At $t=0$, concentration is 0.

As $t \to \infty$, concentration $\to 1$ kg/L.

which is concentration of fluid flowing in.

Snowball 1



"Snowball"

Melting.

- initially 4in in radius
- At time 30 min, radius is 3in.
- When will its radius be 2in?

Need to make an assumption for how snowball melts.

(1) Assume radius changes proportionally with surface area

(2) Volume changes proportionally with surface are.

Let $r(t)$ be radius at time t.

(1) $$\frac{dr}{dt} = -k \cdot \text{Surface area}.$$

$$\boxed{\frac{dr}{dt} = -4\pi k\, r(t)^2}$$

(2) $$\frac{d}{dt}(\text{Volume}) = -k\,(\text{Surface area})$$

$$\frac{d}{dt}\left[\frac{4}{3}\pi r(t)^3\right] = -4\pi k\, r(t)^2$$

$$4\pi r(t)^2 \frac{dr}{dt} = -4\pi k\, r(t)^2$$

$$\boxed{\frac{dr}{dt} = -k}$$

Two competing models for a melting snowball:

(1) $\begin{cases} \dfrac{dr}{dt} = -4\pi k r^2 \\ r(0) = 4 \text{ in} \end{cases}$

(2) $\begin{cases} \dfrac{dr}{dt} = -k \\ r(0) = 4 \text{ in} \end{cases}$

At home:

- Solve both ? $\Rightarrow$ Answer Q.
- ~~what~~ When, mathematically speaking, is the snowball gone in each case?
- think about which model you like better and why.