LECTURE 6

Snowball problem - two competing models.

(1) Volume ↓ $\propto$ S.A.

IVP $\begin{cases} \frac{dr}{dt} = -k \\ r(0) = 4\text{in} \end{cases}$

(2) Radius ↓ $\propto$ S.A.

IVP $\begin{cases} \frac{dr}{dt} = -4\pi k r^2 \\ r(0) = 4\text{ in} \end{cases}$

Solutions

(1) $r(t) = (4 - kt)\text{ in}$

since $r(30) = 3$, $k = 1/30$

$\Rightarrow \boxed{r(t) = 4 - t/30 \text{ in}}$

☆ By separation of vars. OR IF.

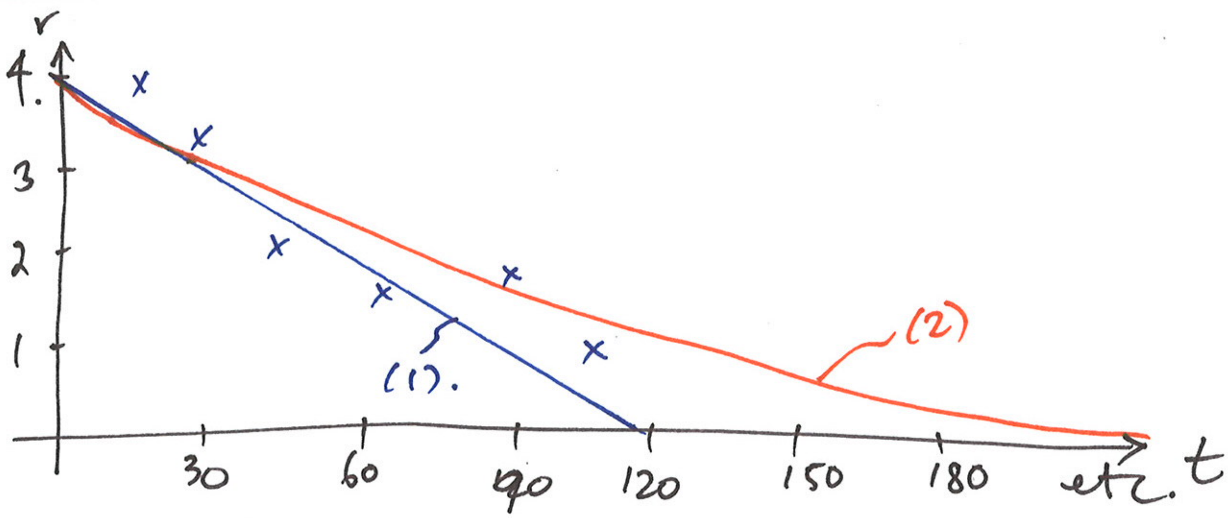
(2) $r(t) = \dfrac{4}{1 + 16\pi kt}$

since $r(30) = 3$, $k = 1/1440\pi$.

$\Rightarrow \boxed{r(t) = \dfrac{360}{t + 90}}$

☆ By separation of variables.

Plot both solutions:



In "real life"

- Fit model (determine k) based on many data points, not just one.
- Prefer model that compares (according to some statistical measurement)

with data.

Always be clear about assumptions.

- heating constant at all times and the same in all directions.
- always spherical!
- etc.

We created a model for snowball melting via 1st order differential eqs.

Second Order Differential Equations:

(intro → 3.1, a little 3.2).

Second order DE - of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \quad \left(\begin{array}{l}\text{highest derivative} \\ \text{is second order}\end{array}\right)$$

LINEAR second order DEs

- if $f(t, y, dy/dt) = g(t) - p(t)\frac{dy}{dt} - q(t)y$.

"standard form"

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t).$$

NOTE: if eq. looks like

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = G(t)$$

recover standard form by $\div P(t)$.

NONLINEAR second order DEs:

As before, contains nonlinearities in "y" terms; can't be put in standard form.

eg) $y\frac{d^2y}{dt^2} + 3y = t$ is nonlinear (b/c of $y\frac{d^2y}{dt^2}$ term).

Note: HARD to solve. Often solved using numerical or graphical means. We'll look at these LATER.

HOMOGENEOUS 2nd order Linear DEs

if $g(t) = 0$.

That is,

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

is a <u>homogeneous</u> 2nd order linear differential eq.

<u>2nd order DE ~~as a~~ from a LINEAR OPERATOR.</u>

An operator <u>maps</u> one function (or vector) onto another.
(☆ VERY loose definition).

Here, our operator is

$$\underbrace{L[\cdot]}_{\text{notation}} = \left(\frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)\right). ☆$$

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y$$

<u>eg)</u> if $p(t) = 0$, $q(t) = 3t$, $y(t) = \cos t$.

$$L[\cos t] = \frac{d^2}{dt^2}(\cos t) + (0)\cdot\frac{d}{dt}(\cos t) + (3t)\cdot\cos t$$

$$= (3t-1)\cos t.$$

$L: \cos t \rightarrow (3t-1)\cos t.$

$\rightarrow L[\cdot] = \left(\frac{d^2}{dt^2} + 3t\right).$

We'll usually see

$$L[y] = 0 \quad \text{or} \quad L[y] = g(t)$$

and be asked to solve for y.

$L[\cdot]$ above (☆) is a LINEAR operator.

i.e. $L[\alpha y_1(t) + \beta y_2(t)]$

$$= \alpha L[y_1(t)] + \beta L[y_2(t)].$$

Showing this: ☆ α, β are constants

$$L[\alpha y_1(t) + \beta y_2(t)] =$$

$$= \boxed{\frac{d^2}{dt^2}\left(\alpha y_1(t) + \beta y_2(t)\right)} + p(t)\frac{d}{dt}\left(\alpha y_1(t) + \beta y_2(t)\right)$$

$$+ q(t)\left(\alpha y_1(t) + \beta y_2(t)\right)$$

$$= \underbrace{\alpha\left\{\frac{d^2 y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right\}}_{\alpha L[y_1]} + \underbrace{\beta\left\{\frac{d^2 y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right\}}_{\beta L[y_2]}$$

$$\frac{d^2}{dt^2}(\alpha y_1 + \beta y_2) = \frac{d^2}{dt^2}(\alpha y_1) + \frac{d^2}{dt^2}(\beta y_2)$$

$$= \alpha \frac{d^2 y_1}{dt^2} + \beta \frac{d^2 y_2}{dt^2}.$$

$$= \alpha L[y_1] + \beta L[y_2].$$

Next couple of weeks:

focus on case where $p(t)$, $q(t)$, or $P(t)$, $Q(t)$, $R(t)$ are <u>constant</u>.

That is, eqs of the form

$$\boxed{a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0}$$

<u>homogeneous</u> eq. w/ constant coefficients.

How are we going to solve these?

By assuming that e^{rt} solves the equation for some constant r, plugging in, and solving for r.

<u>Eg1</u> $y'' - 4y = 0$.

Assume e^{rt} solves this.

~~so y~~

Plug in:

$y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt}$.

$\Rightarrow$ $y'' - 4y = 0$

becomes $(r^2 e^{rt}) - 4(e^{rt}) = 0$

want r. Factor: $e^{rt}(r^2 - 4) = 0$.

since $e^{rt} \neq 0$, $r^2 - 4 = 0$

or $\boxed{r = \pm 2}$

That means both

e^{2t} and e^{-2t}

solve this DE.

But what's the general solution?

$\rightarrow$ Need a theorem, next time.