LECTURE 7

We had $y'' - 4y = 0$.

Posed substitution $y = e^{rt}$, plugged in

Found roots $r = \pm 2$

So both e^{2t}, e^{-2t} solve this DE.

What's the solution?

[Problem session:
Trial Monday 6pm]

Theorem: Principle of superposition.

If $y_1(t)$ and $y_2(t)$ are 2 solutions of $\{ L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$, then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any values of constants c_1 & c_2.

Proof: $L[y_1] = 0$, $L[y_2] = 0$

$$\left.\begin{aligned} L[c_1 y_1 + c_2 y_2] &= L[c_1 y_1] + L[c_2 y_2] \\ &= c_1 L[y_1] + c_2 L[y_2] \end{aligned}\right\} \text{since } L \text{ is LINEAR.}$$

$$= 0.$$ done.

Note: • theorem applies to any second order linear ODE.

- We'll get to showing these are the only solutions soon.

<u>Back to example:</u>

general solution (* show later)

$$y(t) = c_1 e^{2t} + c_2 e^{-2t}.$$

How to get values for c_1 & c_2?

→ Use ICs. Say $y(0) = \cancel{0}.1$

plug in : $y(0) = c_1 + c_2 = \cancel{0}\, 1$

<u>Note</u>: for a 2^{nd} order linear ODE, need <u>2</u> ICs.

Say also $y'(0) = 0$.

$$y'(t) = 2c_1 e^{2t} \cancel{+} -2c_2 e^{-2t}$$

$$y'(0) = 2c_1 - 2c_2 = 0.$$

To obtain constants, solve:

$$\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 - 2c_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1/2 \\ c_2 &= 1/2 \end{aligned}$$

solution is $\boxed{y(t) = \frac{1}{2} e^{2t} + \frac{1}{2} e^{-2t}}$

In general, 2^{nd} order linear ODE $\rightarrow$ 2 ICs.

$$\alpha_1 y(t_0) + \beta_1 y'(t_0) = A$$
$$\alpha_2 y(t_0) + \beta_2 y'(t_0) = B.$$

More generically $\rightarrow$ 2^{nd} order ODE with constant coefficients.

$$ay'' + by' + cy = 0. \quad (*)$$

$y = e^{rt}$
$y' = re^{rt}$
$y'' = r^2 e^{rt}$

Pose substitution $y = e^{rt}$, plug in

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$(ar^2 + br + c)e^{rt} = 0.$$
$$e^{rt} \neq 0 \Rightarrow \boxed{ar^2 + br + c = 0.}$$

- Called the "characteristic equation."
- If r_1 is a root of this equation, then $e^{r_1 t}$ is a solution of $(*)$.

How to solve for r? quadratic eq.!

$$\boxed{r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

We'll go through all the cases for r soon enough
($b^2 - 4ac < 0$, $= 0$, > 0).

For now we'll consider $b^2-4ac>0$
real, distinct roots.

Eg] Solve the IVP

$$\begin{cases} 2y''+y'-y=0 \\ y(0)=1,\ y'(0)=0 \end{cases}$$

Try $e^{rt} \rightarrow$ char. eq. $2r^2+r-1=0$.

$y=e^{rt}$, $y'=re^{rt}$, $y''=r^2e^{rt}$.

in DE: $2\underbrace{(r^2e^{rt})}_{y''}+\underbrace{(re^{rt})}_{y'}-\underbrace{(e^{rt})}_{y}=0$

$$(2r^2+r-1)e^{rt}=0$$

but $e^{rt}\neq 0 \Rightarrow$ must have

$$\underline{2r^2+r-1=0} \quad \text{char. eq.}$$

$$r=\frac{-1\pm\sqrt{(1)^2-4(2)(-1)}}{2(2)}=\tfrac{1}{2},\ -1.$$

$e^{t/2}$, e^{-t} are solutions.

<u>general</u> solution: $y(t)=c_1e^{t/2}+c_2e^{-t}$. &

$y(0)=1 \Rightarrow c_1+c_2=1$

$y'(0)=0 \Rightarrow \frac{1}{2}c_1-c_2=0$

Solving simultaneously, $c_1=2/3$, $c_2=1/3$.

The solution of the IVP is

$$\boxed{y(t) = \frac{2}{3}e^{t/2} + \frac{1}{3}e^{-t}}$$

We've been taking for granted that these two solutions are all there is.

in case of $ay'' + by' + cy = 0$

$y = e^{rt}$, ~~for~~ found roots r_1 + r_2 of char. eq., and said

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

is the general solution.

How do we know that?

More theory for general second order linear DEs

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \quad (*)$$

Theorem: Existence + uniqueness

Consider the IVP $(*)$ with $y(t_0) = y_0$, $y'(t_0) = \tilde{y}_0$ on an open interval I

$[I = -A < t < B)$ that $\boxed{\text{contains } t_0}$ unique.

Then there is exactly one solution $y = \phi(t)$ of this problem, solution exists on interval I.

existence.

Assuming $\boxed{p, q \text{ are continuous}}$ on I.

condition.

Eg| On what interval does a unique solution to the following IVP exist?

$$\begin{cases} y'' + \underbrace{\frac{1}{t}}_{p(t)} y' + \underbrace{t^2}_{q(t)} y = 0 \\ y(1) = 0, \; y'(1) = 1 \end{cases}$$

$q(t) = t^2$ continuous everywhere.

$p(t) = \frac{1}{t}$ continous on $-\infty < t < 0$, $0 < t < \infty$ (not at zero)

initial point $t_0 = 1$ is in interval $0 < t < \infty$.

$\Rightarrow$ $\boxed{\text{Unique solution exists on } 0 < t < \infty}$

Eg $\begin{cases} ay'' + by' + cy = 0 \\ y(t_0) = A, \; y'(t_0) = B \end{cases}$

Where does a unique solution exist?

EVERYWHERE.

"$p(t)$" = b/a, "$q(t)$" = c/a

are constant → cts everywhere.

Next up:

Given that $y_1(t)$ and $y_2(t)$ solve (*),

How can we know or be sure that

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

is the general solution.