LECTURE 8

HW3: p.144 # 1, 9, 13, 23
p.155 # 1

Solutions to HW2 posted later today.

Problem session Monday 6-7pm
Rm: ? MATX 1102 ? (Check on Monday).

$L[y] = y'' + p(t)y' + q(t)y = 0.$ (*).

Say $y_1(t)$, $y_2(t)$ both solve

How can I be sure that the gen. sol.
is $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

First. Let's make sure that you can ALWAYS solve an IVP with this expression.

Take (*) with initial conditions
$y(t_0) = y_0, \quad y'(t_0) = \tilde{y}_0.$

Say $y_1(t)$ & $y_2(t)$ solve (*).

Can you find c_1 & c_2 so that
$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$
satisfies the IVP?

Plug in:

$$y(t_0) = c_1\, y_1(t_0) + c_2\, y_2(t_0) = y_0.$$

$$y'(t_0) = c_1\, y_1'(t_0) + c_2\, y_2'(t_0) = \tilde{y}_0.$$

2 eqs, 2 unknowns.

Write ~~a~~ matrix eq:

$$\underbrace{\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}}_{M.} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ \tilde{y}_0 \end{pmatrix}$$

To solve, invert M. $\Rightarrow$ NEED $\det(M) \neq 0$.

$$W = \det(M)$$
$$= y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)$$

is called the wronskian.

Notation $W[y_1, y_2](t_0)$.

$\therefore$ It is ALWAYS possible to choose constants c_1, c_2 so that

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

satisfies the DE $(*)$ with ICs

$y(t_0) = y_0$, $y'(t_0) = \tilde{y}_0$

IF AND ONLY IF.

$$W[y_1, y_2](t_0) = y_1 y_2' - y_2 y_1' \Big|_{t=t_0}.$$

is NOT zero. (at $t=t_0$).

(Theorem 3.2.3).

OK, so then can we consider

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

to be the general solution?

Yep! (with an additional constraint).

Theorem: Suppose $y_1(t)$ & $y_2(t)$ are solutions to the differential eq.

$$L[y] = y'' + p(t) y' + q(t) y = 0.$$

Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$ with

arbitrary constants c_1 & c_2 includes every solution if and only if ($\Longleftrightarrow$ or iff) there is a point t_0 where

the Wronskian ~~is~~ of y_1 & y_2 is not zero.
(proof: p. 150)

$\Downarrow$

Turns out this means $W \neq 0$ everywhere.

Take-home:

if $W[y_1, y_2] \neq 0$.

→ $y(t) = c_1 y_1(t) + c_2 y_2(t)$ with arbitrary coefficients c_1 & c_2 is the general solution.

→ We call $y_1(t)$ & $y_2(t)$ a FUNDAMENTAL SOLUTION SET.
(can construct a solution for any ICs from them).

Why is Wronskian so key?
→ it tells us if y_1 & y_2 are linearly independent.

Two functions $f_1(x)$ and $f_2(x)$ $(\neq 0)$.
are linearly DEPENDENT. if can choose α_1, α_2 so that $(\alpha_1, \alpha_2 \neq 0)$.

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) = 0.$$

If $y_1(t)$ + $y_2(t)$ that solve (*).
were linearly dependent, then
$\exists$ (there exists) $\alpha_1, \alpha_2 \neq 0$
such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0$$

$$y_2(t) = -\alpha_1/\alpha_2 \, y_1(t).$$

So
$$\begin{aligned} y(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 y_1(t) + c_2\left(-\frac{\alpha_1}{\alpha_2} y_1(t)\right) \\ &= \underbrace{\left(c_1 - \frac{\alpha_1 c_2}{\alpha_2}\right)}_{\text{some const.}} y_1(t) \end{aligned}$$

What we thought was 2 solutions is really just one!

Need $y_1(t)$ + $y_2(t)$ linearly INDEP.

And

if the Wronskian ~~$W[y_1, y_2](t)$~~

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2$$

"belongs to" or "in"

on some interval I $(t \in I)$

is non-zero for some t in that interval, y_1 + y_2 are linearly independent.

Proof: Hint: think about the definition of linear dependence & go from there.

Finally. W is important.

Do I need $y_1(t)$ + $y_2(t)$ to compute it?

No! Abel's theorem.

Let's be clever.

Given $L[y] = y'' + p(t)y'(t) + q(t)y = 0$. (*),

assume p + q are continuous on an interval I.

(so we know unique sol. exists).

Assume $y_1(t)$ & $y_2(t)$ solve (*).

Then
$$y_1'' + p(t)\,y_1' + q(t)\,y_1 = 0 \quad ①$$
$$y_2'' + p(t)\,y_2' + q(t)\,y_2 = 0. \quad ②$$

Recall $W = y_1 y_2' - y_1' y_2$

$-y_2(t)\cdot ① + y_1(t)\, ②$

$$\Rightarrow \underbrace{y_2'' y_1 - y_1'' y_2}_{\text{can I write this differently}} + p(t)\underbrace{[y_1 y_2' - y_1' y_2]}_{=W} = 0.$$

$$W' = y_2'' y_1 - y_1'' y_2 \;!$$

$$\therefore \quad W'(t) + p(t)\,W(t) = 0.$$

use integrating factor $\mu(t) = e^{\int^t p(t')dt'}$

to obtain:

$$\boxed{W[y_1, y_2](t) = C\, e^{-\int^t p(t')dt'}}$$

Note: this means that

$W = 0$ for all t in I.

(if $C = 0$) OR is NEVER

zero (if $C \neq 0$).

↳ as $e^x \neq 0$ for any x.

Take home on 2^{nd} order linear ODEs:

For $L[y] = y'' + p(t)y' + q(t)y = 0$

→ 2 linearly independent sol~~es~~s.

(check $W[y_1, y_2] \neq 0$).

THEN → general solution linear combo

$$c_1 y_1(t) + c_2 y_2(t)$$

THEN → To get a solution to IVP you need 2 ICs.