LECTURE 9

Problem session → tonight, 6pm
MATX 1102

Practice midterm available on website.

From last time - want to prove nonzero Wronskian guarantees linear independence.

Take $y_1(t)$, $y_2(t)$ (differentiable on some interval).

If they're linearly <u>dependent</u> then we can find constants $\alpha, \beta \neq 0$ such that:

$$\alpha y_1(t) + \beta y_2(t) = 0.$$

Take der → $\alpha y_1'(t) + \beta y_2'(t) = 0$

Matrix form:
$$\underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{M \text{ (matrix)}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det(M) = W[y_1, y_2](t).$$

- If $W \neq 0$ ($\det(M) \neq 0$), M <u>IS</u> invertible.

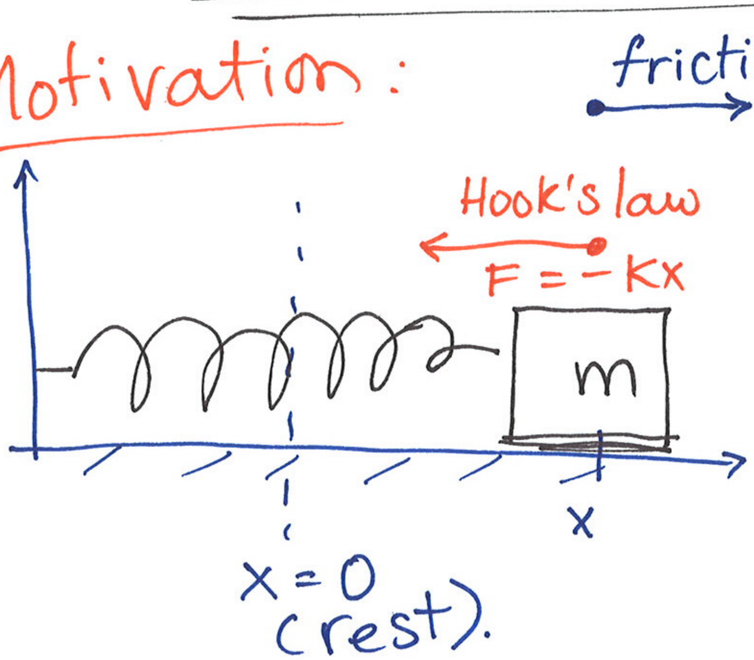
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \alpha = 0, \; \beta = 0$$

so if $W \neq 0$, linearly indep. $y_1(t)$ & $y_2(t)$.

- If $W=0$ ($\det(M)=0$), M is not invertible, CAN find $\alpha, \beta \neq 0$ that satisfies matrix eq.
  $\Rightarrow y_1, y_2$ lin. dep.

Second order ODEs w/ CONSTANT coefficients

Motivation:



Force balance

$$F = -kx - bv.$$

$$ma = -kx - bv$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}.$$

$$\begin{cases} m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = 0 \\ x(0) = \text{initial position} \\ x'(0) = \text{initial velocity} \end{cases}$$

2nd order linear DEs w/ constant coefs used to model oscillator... model

- pendulums
- tall blds
- bridge suspension
- circuits (LCR)

etc.

Keep this in mind...

General case $ay'' + by' + cy = 0$.

Pose sub $y = e^{rt}$, plug in

$$(y' = re^{rt}, \quad y'' = r^2 e^{rt})$$

$$\Rightarrow \quad ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

$e^{rt} \neq 0$, $\div e^{rt}$:

$$\boxed{ar^2 + br + c = 0}$$

CHARACTERISTIC EQ.

Char. eq. has roots:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$r_1 \rightarrow$ +ve root

$r_2 \rightarrow$ -ve root.

sign of discriminant b^2-4ac affect solution.

Case I: $b^2-4ac<0$ (3.5 in text).

roots are distinct but complex.

$$r_{1,2}=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b}{2a}\pm\frac{\sqrt{-1|b^2-4ac|}}{2a}$$

$$=\underbrace{\frac{-b}{2a}}_{\alpha,\text{ real part}}\pm i\underbrace{\frac{\sqrt{|b^2-4ac|}}{2a}}_{\beta,\text{ imag. part}}\qquad(\sqrt{-1}=i).$$

$r_{1,2}=\alpha\pm i\beta.$

What do we get? What does the general sol'n look like?

$$y(t)=d_1e^{r_1t}+d_2e^{r_2t}$$

$$=d_1e^{(\alpha+i\beta)t}+d_2e^{(\alpha-i\beta)t}.$$

We'd like a real-valued solution...

De Moivre's Thm

$$e^{i\varphi}=\cos\varphi+i\sin\varphi.$$

or

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}, \quad \cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}.$$

Aside:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi. \quad (1)$$

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi \quad (2)$$

$$(1) + (2) \Rightarrow e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi$$

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}.$$

$$(1) - (2) \Rightarrow \sin\varphi = \ldots$$

$$y(t) = d_1 e^{\alpha t} e^{i\beta t} + d_2 e^{\alpha t} e^{-i\beta t}$$

use de Moivre's (or other), re-arrange and obtain.

$$\boxed{y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)}$$

is ~~a~~ the general solution.

real part of root

imag. part of root.

How can I be sure that

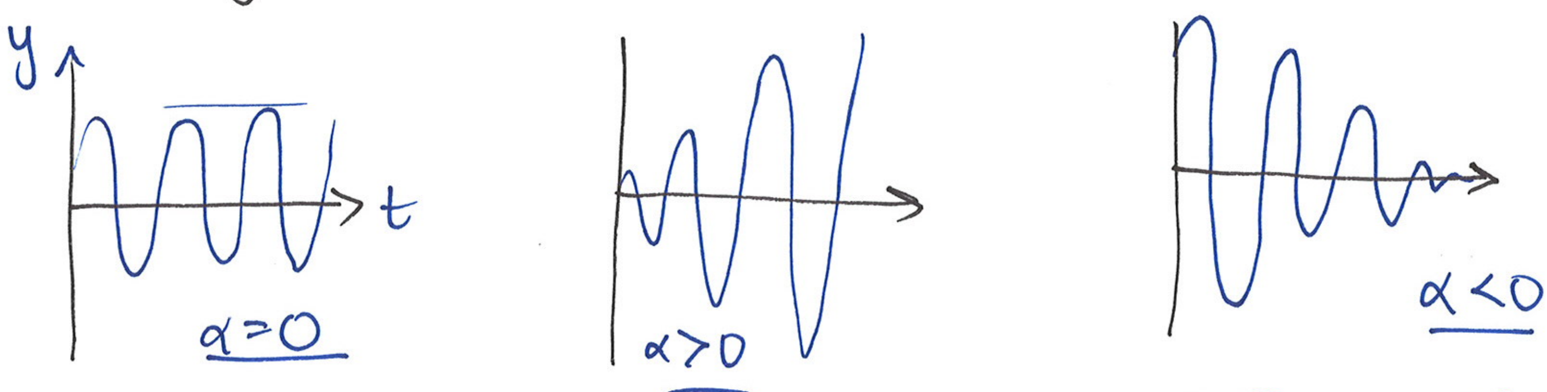
$$y_1(t) = e^{\alpha t}\cos\beta t$$
$$y_2(t) = e^{\alpha t}\sin\beta t$$

are the 2 linearly independent solutions?

$\Rightarrow$ check Wronskian.

<u>Behaviour of Solutions:</u>

$$y(t) = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t.$$



Recall $\alpha = \dfrac{-b}{2a} = \dfrac{-b}{2m}$. $(m>0)$ } For mass-spring.

undamped [$\alpha = 0 \rightsquigarrow$ no damping. $(b=0)$

damped [$\alpha < 0 \rightsquigarrow b > 0$, coeff. of damping > 0
damped oscillations.

$\alpha > 0 \rightsquigarrow$ doesn't make sense physically.

Case II $b^2 - 4ac > 0$.

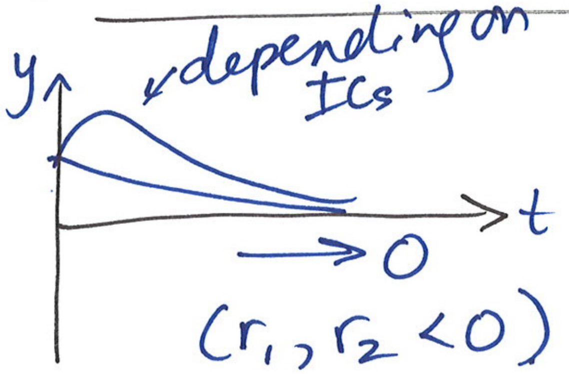
roots $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

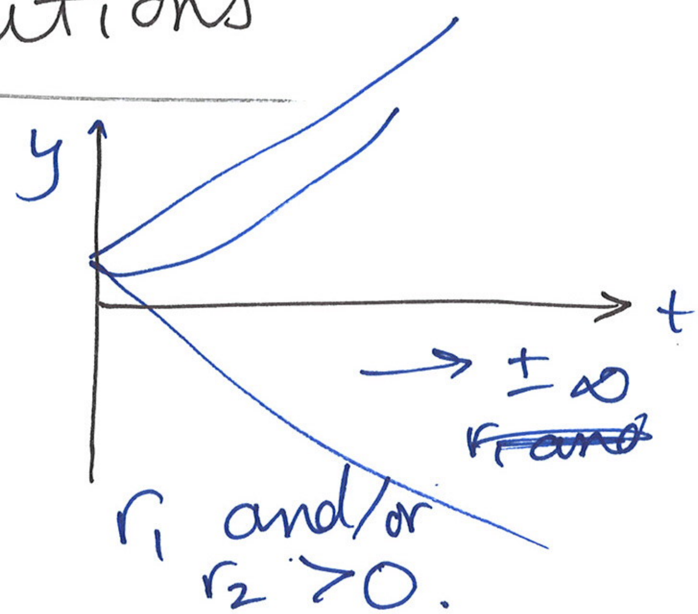
~~rea~~ distinct and real (last week).

gen. solution is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Behaviour of solutions





$\Downarrow$

relevant for mass-spring case, damping coef > 0.

OVERDAMPED.

(damping $b^2 > 4mk$ in case of mass-spring system)